Annales de la faculté des sciences de Toulouse

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Annales de la faculté des sciences de Toulouse 6e série, tome 14, n° 3 (2005), p. 435-458

<http://www.numdam.org/item?id=AFST_2005_6_14_3_435_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIV, n° 3, 2005
pp. 435–458

On the relation between the Borel sum and the classical solution of the Cauchy problem for certain partial differential equations (*)

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ABSTRACT. — For a divergent solution of the Cauchy problem of a non-Kowalevskian equation such as the heat equation or the Airy equation or the Beam equation, the condition for the k-summability or the Borel summability was obtained by Miyake [Miy] and the integral representation of the Borel sum was obtained by Ichinobe [Ich 1] (see Ichinobe [Ich 2] for more detail). By the results of Ichinobe, we know that for such an equation except the heat equation the Borel sum is far or different from the classical solution which can be obtained by the theory of Fourier integrals.
In this paper, we shall show the manner how the integral representation of the classical solution is derived from that of the Borel sum by deforming the paths of integrations, which may be regarded as a decomposition of the fundamental solution in the real Euclidean planes into the complex planes.

RÉSUMÉ. — Pour la solution divergente du problème de Cauchy pour une équation non Kowalevskienne telle que l'équation de la chaleur ou l'équation d'Airy ou encore l'équation de Beam, la condition pour la k-sommabilité ou la sommabilité de Borel a été obtenue par Miyake [Miy] et la représentation intégrale de la somme de Borel a été obtenue par Ichinobe [Ich 1] (voir Ichinobe [Ich 2] pour le détail).
Par les résultats d'Ichinobe, nous savons que, pour une telle équation exceptée l'équation de la chaleur, la somme de Borel est loin ou différente de la solution classique qui peut être obtenue par la théorie des intégrales de Fourier.
Dans cette note, nous montrons comment la représentation intégrale de la solution classique est dérivée de celle de la somme de Borel par déformation des chemins d'intégration, qui peut être considérée comme la formule de décomposition de la solution fondamentale de l'équation aux plans Euclidiens réels dans les plans complexes.

(*) Reçu le 15 octobre 2003, accepté le 12 janvier 2005
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1. Introduction

We consider the following two Cauchy problems for equations of non-Kowalevski type

$$(\mathbf{CP})_{\mathbb{R}} \qquad \partial_t u(t,x) = \alpha \partial_x^q u(t,x), \qquad u(0,x) = \varphi(x) \quad (t>0,\ x\in\mathbb{R}),$$
$$(\mathbf{CP})_{\mathbb{C}} \qquad \partial_\tau u(\tau,z) = \alpha \partial_z^q u(\tau,z), \qquad u(0,z) = \varphi(z) \quad ((\tau,z)\in\mathbb{C}^2),$$

where $q \geqslant 2$, $\alpha = 1$ ($\arg\alpha = 0$) when $q \equiv 2,3 \pmod 4$ and $\alpha = -1$ ($\arg\alpha = \pi$) when $q \equiv 0,1 \pmod 4$. These assumptions for α are only for the simplicity of the statement of our result. Especially, under these conditions for α, the Cauchy problem $(\mathbf{CP})_{\mathbb{R}}$ is uniquely solvable in the class $\mathcal{S}$ of Schwartz' rapidly decreasing functions in x variable.

In this paper, we shall give a relationship between the "Classical solution" of $(\mathbf{CP})_{\mathbb{R}}$ and the "Borel sum" of the divergent solution of $(\mathbf{CP})_{\mathbb{C}}$, where the Cauchy data $\varphi(z)$ is assumed to be holomorphic in a neighbourhood of the origin. Precisely, we shall show that the "Classical solution" of $(\mathbf{CP})_{\mathbb{R}}$ is derived from a deformation of paths in the integral representation of the "Borel sum" in 0 direction under some conditions for the Cauchy data $\varphi(z)$ of $(\mathbf{CP})_{\mathbb{C}}$.

First, we give the definition of the "Classical solution" of $(\mathbf{CP})_{\mathbb{R}}$. Let the Cauchy data $\varphi(x)$ be taken from $\mathcal{S}$. Then the unique solution $u_c(t,x)$ in $\mathcal{S}$ is given by

$$u_c(t,x) = \int_{-\infty}^{+\infty} \varphi(x+y) E(t,y;q,\alpha)dy, \quad t>0,\ x\in\mathbb{R}. \tag{1.1}$$

Here the kernel function $E(t,y;q,\alpha)$ is given by

$$E(t,y;q,\alpha) = \frac{1}{(qt)^{1/q}} \widetilde{E}_{q,\alpha}\left(\frac{y}{(qt)^{1/q}}\right), \tag{1.2}$$

with the function $\widetilde{E}_{q,\alpha}(z)$ given by

$$\widetilde{E}_{q,\alpha}(z) = \frac{1}{2\pi i}\int_\gamma \exp\left(zs + \alpha\frac{(-s)^q}{q}\right) ds, \quad z\in\mathbb{C}, \tag{1.3}$$

where the path of integration γ is given as follows.

(I) When q is even, the path γ runs from $-i\infty$ to $+i\infty$.

(II) When q is odd, the path γ is any curve which begins at ∞ in the sector $3\pi/2 - \pi/q < \arg s < 3\pi/2$ and ends at ∞ in the sector $\pi/2 < \arg s < \pi/2 + \pi/q$.

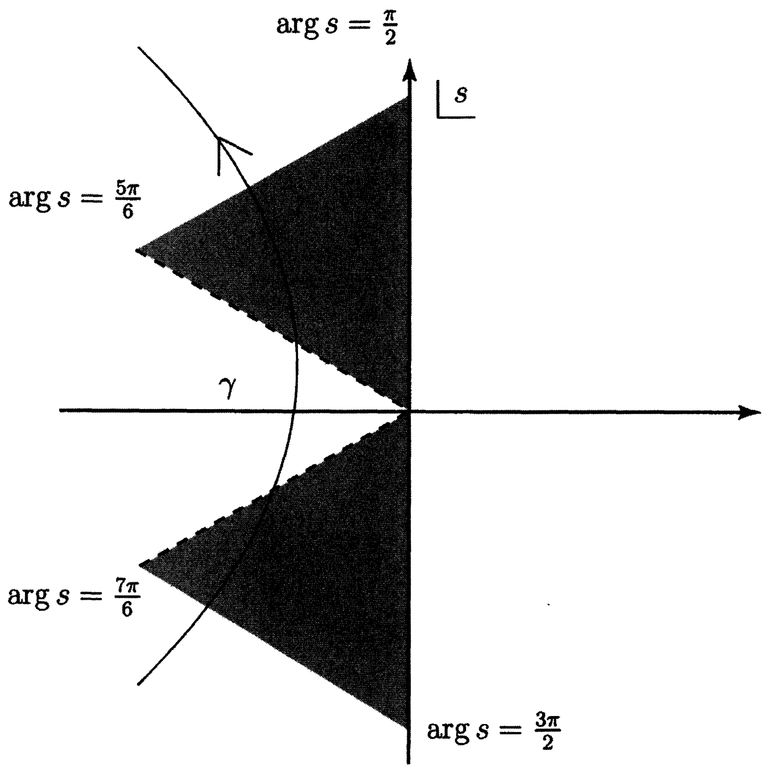

Figure 1. — The case $q = 3$

We remark that the kernel function $\widetilde{E}_{q,\alpha}(z)$ has the following asymptotic estimates on the real axis as $z \to \pm\infty$ (see (3.20) and (3.21)).

(I) When q is even,

$$|\widetilde{E}_{q,\alpha}(z)| \leqslant C \exp\left\{-c|z|^{q/(q-1)}\right\} |z|^{q/2(q-1)-1} \quad \text{as } z \to \pm\infty \ (C, c > 0). \tag{1.4}$$

(II) When q is odd,

$$|\widetilde{E}_{q,\alpha}(z)| \leqslant \begin{cases} C \exp\left\{-c|z|^{q/(q-1)}\right\} |z|^{q/2(q-1)-1} & \text{as } z \to +\infty \ (C, c > 0), \\ C|z|^{q/2(q-1)-1} \quad \text{as } z \to -\infty \quad (C > 0). \end{cases} \tag{1.5}$$

From these asymptotic estimates, we see that the integral formula (1.1) of the solution does work in a wider class than $\mathcal{S}$ for the Cauchy data $\varphi(x)$ (see Remark 3.3).

We call the solution of $(\mathbf{CP})_{\mathbb{R}}$, which is given by the integral formula (1.1), the "Classical solution" of $(\mathbf{CP})_{\mathbb{R}}$.

Next, we give the "Borel sum" of the divergent solution of $(\mathbf{CP})_{\mathbb{C}}$.

The Cauchy problem $(\mathbf{CP})_{\mathbb{C}}$ has a unique formal solution

$$\hat{u}(\tau,z)=\sum_{n\geqslant 0}\alpha^n\varphi^{(qn)}(z)\frac{\tau^n}{n!}\underset{\text{put}}{=}\sum_{n\geqslant 0}u_n(z)\tau^n. \tag{1.6}$$

By Cauchy's integral formula, we see that the coefficients $u_n(z)$ have the following estimates

$$\max_{|z|\leqslant r_1}|u_n(z)|\leqslant CK^n((q-1)n)!,\quad n=0,1,2,\ldots, \tag{1.7}$$

by some positive constants r_1, C and K. From the assumption that $q\geqslant 2$, this formal solution $\hat{u}(\tau,z)$ is divergent with respect to τ-variable.

In order to state the Borel summability of the divergent solution and to give the integral representation of the Borel sum, we need some definitions (cf. [Bal]).

For $d\in\mathbb{R}$, $\beta>0$ and $\rho\,(0<\rho\leqslant\infty)$, a sector $S(d,\beta,\rho)$ is defined by

$$S(d,\beta,\rho):=\left\{\tau\in\mathbb{C};|\arg\tau-d|<\frac{\beta}{2},0<|\tau|<\rho\right\}, \tag{1.8}$$

and d, β and ρ are called the direction, the opening angle and the radius of this sector, respectively.

Let $\hat{u}(\tau,z)$ be the divergent solution (1.6) of $(\mathbf{CP})_{\mathbb{C}}$, and $u(\tau,z)$ be an analytic function on $S(d,\beta,\rho)\times B(r_2)$, $B(r_2):=\{z\in\mathbb{C};|z|\leqslant r_2\}$. Then we say that $u(\tau,z)$ has an asymptotic expansion $\hat{u}(\tau,z)$ of Gevrey order $1/(q-1)$ in $S(d,\beta,\rho)$, if for any relatively compact subsector S' of $S(d,\beta,\rho)$, there exists $r_3(\leqslant\min\{r_1,r_2\})$ such that for any N, we have

$$\max_{|z|\leqslant r_3}\left|u(\tau,z)-\sum_{n=0}^{N-1}u_n(z)\tau^n\right|\leqslant CK^N\left((q-1)N\right)!|\tau|^N,\quad \tau\in S', \tag{1.9}$$

by some positive constants C and K. In this case, if $u(\tau,z)$ is a solution of the equation of $(\mathbf{CP})_{\mathbb{C}}$, $u(\tau,z)$ is called an asymptotic solution of $\hat{u}(\tau,z)$ of Gevrey order $1/(q-1)$ in $S(d,\beta,\rho)$.

When the opening angle β is less than $(q-1)\pi$, there always exist infinitely many asymptotic solutions $u(\tau, z)$ for any direction $d \in \mathbb{R}$ and any Cauchy data $\varphi(z)$ which is holomorphic in a neighbourhood of the origin (cf. [LMS]).

When $\beta > (q-1)\pi$ for the opening angle β, there does not exist such an asymptotic solution without any condition for the Cauchy data $\varphi(z)$ (see Theorem 1.1 below). But if such asymptotic solutions exist, then it is unique. In this sense, such an asymptotic solution $u(\tau, z)$ is called the *Borel sum of* $\hat{u}(\tau, z)$ in d direction. We write it by $u_B^d(\tau, z)$, and we say that $\hat{u}(\tau, z)$ is *Borel summable* in d direction.

Now, we give a theorem for the Borel summability which is a special case in Miyake's paper [Miy].

THEOREM 1.1 (MIYAKE). — *The formal solution $\hat{u}(\tau, z)$ of $(\mathbf{CP})_{\mathbb{C}}$ is Borel summable in d direction if and only if there exists a positive constant ε such that*

(1) *the Cauchy data φ can be continued analytically in a domain*

$$\Omega_\varepsilon(d; q, \alpha) := \bigcup_{m=0}^{q-1} S\left(\frac{d + \arg\alpha + 2\pi m}{q}, \varepsilon, \infty\right), \tag{1.10}$$

(see below Figure 2),

(2) *the Cauchy data φ has a growth condition of exponential order at most $q/(q-1)$ in $\Omega_\varepsilon(d; q, \alpha)$, that is, there exist positive constants C and γ such that the following growth estimate holds.*

$$|\varphi(z)| \leqslant C \exp\left(\gamma |z|^{q/(q-1)}\right), \quad z \in \Omega_\varepsilon(d; q, \alpha). \tag{1.11}$$

In order to give the explicit formula of the Borel sum, we need a preparation of the Meijer G-function.

The Meijer G-Function. (cf. [MS, p. 2]) For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q) \in \mathbb{C}^q$ with $\alpha_\ell - \gamma_j \in \mathbb{N}$ $(\ell = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ such that $0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant q$, we define

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{matrix}\right) = \frac{1}{2\pi i} \int_I \frac{\prod_{j=1}^m \Gamma(\gamma_j + s) \prod_{\ell=1}^n \Gamma(1 - \alpha_\ell - s)}{\prod_{j=m+1}^q \Gamma(1 - \gamma_j - s) \prod_{\ell=n+1}^p \Gamma(\alpha_\ell + s)} z^{-s} ds, \tag{1.12}$$

where the path of integration I which runs from $\kappa - i\infty$ to $\kappa + i\infty$ for any fixed $\kappa \in \mathbb{R}$ is taken as follows; all poles of $\Gamma(\gamma_j + s)$, $\{-\gamma_j - k\,;\, k \geqslant 0, j = 1, 2, \dots, m\}$, lie to the left of I and all poles of $\Gamma(1 - \alpha_\ell - s)$, $\{1 - \alpha_\ell + k\,;\, k \geqslant 0, \ell = 1, 2, \dots, n\}$, lie to the right of I, which is possible by the conditions that $\alpha_\ell - \gamma_j \in \mathbb{N}$.

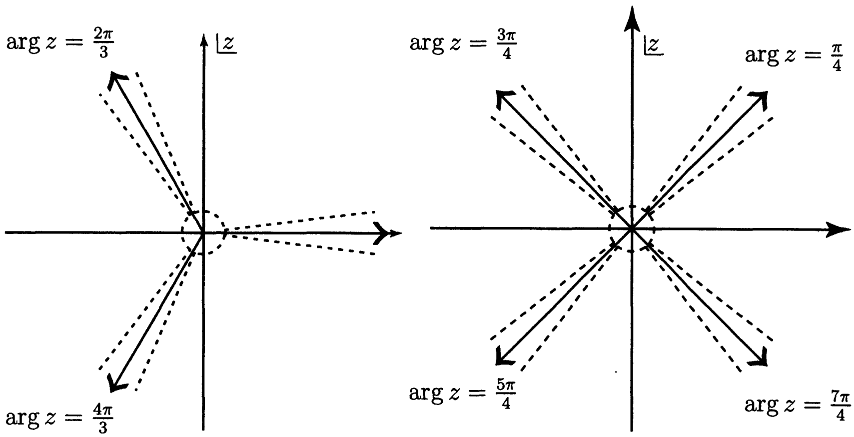

Figure 2. — $\Omega_\varepsilon(0,3,1)$ and $\Omega_\varepsilon(0,4,-1)$

In the following, the integration $\int_0^{\infty(\theta)}$ denotes the integration from 0 to ∞ along the half line of argument θ, and we use the following notations.

$$\boldsymbol{q} = (1, 2, \dots, q), \quad \frac{\boldsymbol{q}}{q} = \left(\frac{1}{q}, \frac{2}{q}, \dots, \frac{q}{q}\right),$$

and $\widehat{\boldsymbol{q}_j}$ is the $(q-1)$-ple of integers, which is obtained by omitting j-th component from $\boldsymbol{q}$.

Next, we give a theorem for the integral representation of the Borel sum in 0 direction, which is a special case in the author's paper [Ich 1, 2].

THEOREM 1.2 (ICHINOBE). — *Under the conditions* (1) *and* (2) *in Theorem* 1.1, *the Borel sum* $u_B^0(\tau, z)$ *is obtained by the analytic continuation of the following function*

$$u_B^0(\tau, z) = \sum_{m=0}^{q-1} \int_0^{\infty((\arg\alpha + 2\pi m)/q)} \varphi(z+\zeta) k_q(\tau, \zeta\omega_q^{-m}; \alpha)\omega_q^{-m} d\zeta, \quad (1.13)$$

where $(\tau, z) \in S(0, \beta, \rho) \times B(r)$ *with* $\beta < (q-1)\pi$, $\omega_q = \exp(2\pi i/q)$, *and the kernel function* $k_q(\tau, \zeta; \alpha)$ *is given by*

$$k_q(\tau, \zeta; \alpha) = \frac{C_q}{\zeta} G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \widehat{\boldsymbol{q}_q}/q\right), \quad Z_\alpha = \frac{1}{q^q \alpha} \frac{\zeta^q}{\tau}, \tag{1.14}$$

with

$$C_q = \frac{1}{\prod_{j=1}^{q-1} \Gamma(j/q)}.$$

In special cases the kernel function is given explicitly (cf.[LMS], [Ich 1]).

PROPOSITION 1.3. —

(a) *The case* $(q, \alpha) = (2, 1)$, *that is, the case of the heat equation. The kernel function is given by*

$$k_2(\tau, \zeta; 1) = \frac{1}{\sqrt{4\pi\tau}} e^{-\zeta^2/4\tau}. \tag{1.15}$$

(b) *The case* $(q, \alpha) = (3, 1)$, *that is, the case of the Airy equation. The kernel function is given by*

$$k_3(\tau, \zeta; 1) = \frac{1}{(3\tau)^{1/3}} \mathrm{Ai}\left(\frac{\zeta}{(3\tau)^{1/3}}\right). \tag{1.16}$$

Here Ai *denotes the Airy function which is defined by*

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_\gamma \exp\left(zs - \frac{s^3}{3}\right) ds, \tag{1.17}$$

where the path γ *is given by the same one as in* (1.3) *with* $q = 3$ (*and* $\alpha = 1$).

The statement (**a**) was given by [LMS] and the statement (**b**) was given by [Ich 1].

When $q = 2$, the kernel function for the Borel sum is given by the heat kernel. Therefore, the integral representation (1.13) of the Borel sum just coincides with (1.1) that of the Classical solution. On the other hand, when $q \geqslant 3$, the integral representations (1.1) and (1.13) are completely different. Therefore, our interest in this paper is to ask the relationship between the integral representations (1.1) and (1.13) when $q \geqslant 3$.

2. Main result

We can give the relationship between the integral representations (1.1) and (1.13) as follows.

THEOREM 2.1. — *Under the additional conditions for the Cauchy data $\varphi(z)$ which are stated below, the integral representation* (1.1) *is obtained by deforming the paths of integrations in* (1.13) *as the following manner. We divide q rays of integrations in the representation* (1.13) *into two groups, R_+ and R_-. Here R_+ (resp. R_-) denotes the group of the rays which are in the right (resp. left) half plane of the complex plane. Then all the integrations along the rays in R_+ (resp. R_-) can be changed into the integration on the positive (resp. negative) real axis.*

(I) (*Generalization of the heat equation*) *When q is even, the Cauchy data $\varphi(z)$ can be continued analytically in two sectors $\Delta_q = S(0, \pi - 2\pi/q, \infty) \cup S(\pi, \pi - 2\pi/q, \infty)$ with the same growth condition as in the Borel summability in Theorem* 1.1. *We remark that $\Delta_2 = \phi$ $(q = 2)$ by the definition, and in this case it is not necessary to assume additional condition.*

(II) (*Generalization of the Airy equation*) *When q is odd, we define $\Delta_q = S(0, \pi - 3\pi/q, \infty) \cup S(\pi, \pi - \pi/q, \infty)$ for $q > 3$, and $\Delta_3 = S(\pi, 2\pi/3, \infty)$ for $q = 3$. We assume that the Cauchy data $\varphi(z)$ can be continued analytically in Δ_q with the same growth condition as in the Borel summability in Theorem* 1.1. *Further, we assume that there exists a positive constant δ such that, in the region $S(\pi, \delta, \infty)$, $\varphi(z)$ has the following decreasing condition of polynomial order*

$$|\varphi(z)| \leqslant \frac{C}{|z|^{q/2(q-1)+\lambda}}, \quad z \in S(\pi, \delta, \infty), \tag{2.1}$$

by some positive constants C and λ.

Remark 2.2. — When q is even, the condition for α corresponds to that the equation is of parabolic type in t positive direction like the heat equation. Let us consider the Schrödinger type equation, that is, the equation is given by

$$i\partial_t u(t,x) = (i\partial_x)^q u(t,x), \quad t > 0, \ x \in \mathbb{R},$$

with an even integer q. Then we can give the similar result to Theorem 2.1, but in this case, we ask the same decreasing condition as (2.1) for the

Cauchy data in two sectors $S(0,\delta,\infty)\cup S(\pi,\delta,\infty)$. We do not give the proof of this statement, since it is done by the similar way with that of Theorem 2.1 (II).

3. The proof of Theorem 2.1

Before we give the proof of Theorem 2.1, we introduce the auxiliary functions.

We first define the integral paths of the auxiliary functions. For that purpose, we divide the complex plane into $2q$ sectors whose directions are $j\pi/q \pmod{2\pi}$ $(j = 0, 1, \dots, 2q-1)$ and their opening angles are π/q for all. We name q sectors among these as follows. For any integer j, we define sectors S_j by

$$S_j = \begin{cases} S\left(\dfrac{2(j-1)}{q}\pi, \dfrac{\pi}{q}, \infty\right), & q \equiv 0, 3 \pmod 4, \\ S\left(\dfrac{2j-1}{q}\pi, \dfrac{\pi}{q}, \infty\right), & q \equiv 1 \pmod 4, \\ S\left(\dfrac{2j-3}{q}\pi, \dfrac{\pi}{q}, \infty\right), & q \equiv 2 \pmod 4. \end{cases} \tag{3.1}$$

(See below Figure 3). Note that $S_j = S_{j+q}$ for any integer j.

Then for any integer j, the path γ_j is defined by any curve which begins at ∞ in the sector S_{j+1} and ends at ∞ in the sector S_j (see below Figure 3).

By employing these paths, we define the auxiliary functions by the following form.

$$v_j(z) = v_j(z;q,\alpha) = \frac{1}{2\pi i}\int_{\gamma_j} \exp\left(zs + \alpha\frac{(-s)^q}{q}\right) ds, \ \ z \in \mathbb{C}, \tag{3.2}$$

for any integer j. Since $S_j = S_{j+q}$, we have $v_j = v_{j+q}$.

We define an another auxiliary function

$$w(z) = w(z;q,\alpha) = \frac{1}{2\pi i}\int_{\gamma} \exp\left(zs + \alpha\frac{(-s)^q}{q}\right) ds, \quad z \in \mathbb{C}, \tag{3.3}$$

where the path γ is given as follows.

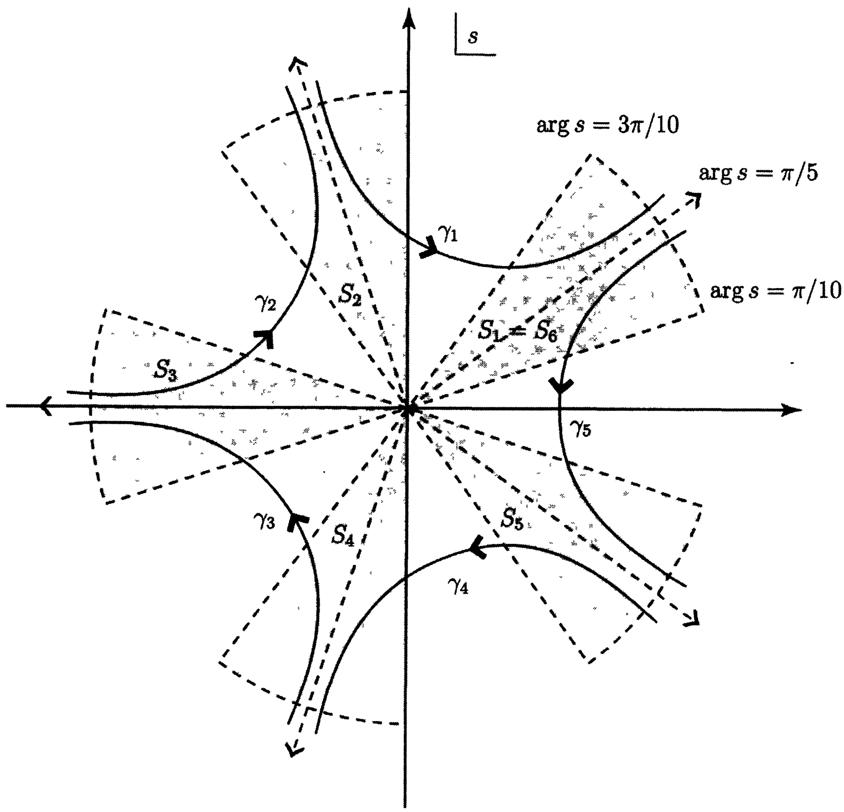

Figure 3. — Sectors S_j's and paths γ_j's in the case $q = 5$

(I) When q is even, the path γ runs from $-i\infty$ to $+i\infty$.

(II) When q is odd, the path γ is any curve which begins at ∞ in the sector $S(3\pi/2 - \pi/(2q), \pi/q, \infty)$ and ends at ∞ in the sector $S(\pi/2 + \pi/(2q), \pi/q, \infty)$.

We summarize the important properties and relations between $\{v_j(z)\}$ and $w(z)$ as a proposition without proofs.

PROPOSITION 3.1. —

(a) *The functions v_j's and w satisfy the following ordinary differential equation*

$$\frac{d^{q-1}y}{dz^{q-1}} + \alpha(-1)^q zy = 0. \tag{3.4}$$

(b) *There are the following relations between* $\{v_j(z)\}$. *For any integer* j

(V1) $\quad v_{j+1}(z) = v_j(z\omega_q)\omega_q, \quad \omega_q = e^{2\pi i/q}.$

(c) *There are the following relations between* $\{v_j(z)\}$ *and* $w(z)$.

(i) *When* $q \equiv 0, 1 \pmod 4$, *we put* $q = 4n,\ 4n+1$ $(n \geqslant 1)$. *Then we have*

$$(\mathbf{V2})_{\text{(i)}} \quad v_{n+1} + \cdots + v_{3n} = w = -(v_1 + \cdots + v_n + v_{3n+1} + \cdots + v_q).$$

(ii) *When* $q \equiv 2, 3 \pmod 4$, *we put* $q = 4n-2$ $(n \geqslant 2)$, $4n-1$ $(n \geqslant 1)$. *Then we have*

$$(\mathbf{V2})_{\text{(ii)}} \quad v_{n+1} + \cdots + v_{3n-1} = w = -(v_1 + \cdots + v_n + v_{3n} + \cdots + v_q).$$

Next, we can prove the following.

PROPOSITION 3.2. —

(a) *Let* $\widetilde{E}_{q,\alpha}(z)$ *be the kernel function of the Classical solution, which is given by* (1.3). *Then we have*

$$\widetilde{E}_{q,\alpha}(z) = w(z; q, \alpha). \tag{3.5}$$

(b) *Let* $k_q(\tau, \zeta; \alpha)$ *be the kernel function of the Borel sum, which is given by* (1.14). *Then we have*

$$k_q(\tau, \zeta; \alpha) = \frac{1}{(q\tau)^{1/q}}\, v_{2n}\left(\frac{\zeta}{(q\tau)^{1/q}}; q, \alpha\right). \tag{3.6}$$

The statement (a) follows from the definitions of two functions. The proof of the statement (b) will be given in the next section.

Proof of Theorem 2.1. — By Proposition 3.2, we have the following expressions.

$$u_c(t,x) = \frac{1}{(qt)^{1/q}} \int_{-\infty}^{+\infty} \varphi(x+y) w\left(\frac{y}{(qt)^{1/q}}\right) dy, \quad (t > 0,\ x \in \mathbb{R}). \tag{3.7}$$

$$u_B^0(\tau, z) = \frac{1}{(q\tau)^{1/q}} \sum_{m=0}^{q-1} \int_0^{\infty((\arg\alpha + 2\pi m)/q)} \varphi(z+\zeta) v_{2n}\left(X\omega_q^{-m}\right)\omega_q^{-m} d\zeta, \tag{3.8}$$

where $(\tau, z) \in S(0, \beta, \rho) \times B(r)$ with $\beta < (q-1)\pi$ and

$$X = \frac{\zeta}{(q\tau)^{1/q}}. \tag{3.9}$$

By using the functional equalities (**V1**), the Borel sum $u_B^0(\tau, z)$ is rewritten in the form

$$u_B^0(\tau, z) = \frac{1}{(q\tau)^{1/q}} \sum_{m=0}^{q-1} \int_0^{\infty((\arg \alpha + 2\pi m)/q)} \varphi(z+\zeta) v_{2n-m}(X) d\zeta. \tag{3.10}$$

We fix $\tau = t > 0$.

When the Cauchy data $\varphi(z)$ satisfies the conditions (I) or (II) in Theorem 2.1, we can prove the following formula.

$$\begin{aligned} &\int_0^{\infty((\arg \alpha + 2\pi m)/q)} \varphi(z+\zeta) v_{2n-m}(X) d\zeta \\ &= \begin{cases} \int_0^{+\infty} \varphi(z+\zeta) v_{2n-m}(X) d\zeta & \text{if the ray is in } R_+, \\ \int_0^{-\infty} \varphi(z+\zeta) v_{2n-m}(X) d\zeta & \text{if the ray is in } R_-, \end{cases} \end{aligned} \tag{3.11}$$

where

$$X = \frac{\zeta}{(qt)^{1/q}}.$$

For a while, by assuming this formula (3.11), we prove our theorem.

• (i)-1. The case $q = 4n$ $(n \geqslant 1)$. Since the rays of integrations in (3.10) with $m = 0, \dots, n-1, 3n, \dots, q-1$ (resp. $m = n, \dots, 3n-1$) belong to R_+ (resp. R_-), we have

$$\begin{aligned} (*)_{(\mathrm{i})-1} \quad & u_B^0(t, z) \\ = & \frac{1}{(qt)^{1/q}} \left[\int_0^{+\infty} \varphi(z+\zeta) \left\{ v_{2n} + \cdots + v_{2n-(n-1)} \right.\right. \\ & \left. + v_{2n-(3n)} + \cdots + v_{2n-(q-1)} \right\} (X) d\zeta \\ & \left. + \int_0^{-\infty} \varphi(z+\zeta) \left\{ v_{2n-n} + \cdots + v_{2n-(3n-1)} \right\} (X) d\zeta \right] \end{aligned}$$

$$= \frac{1}{(qt)^{1/q}} \left[\int_0^{+\infty} \varphi(z+\zeta) \left\{ v_{2n} + \cdots + v_{n+1} + v_{3n} + \cdots + v_{2n+1} \right\} (X) d\zeta \right.$$
$$\left. + \int_0^{-\infty} \varphi(z+\zeta) \left\{ v_n + \cdots + v_1 + v_q + \cdots + v_{3n+1} \right\} (X) d\zeta \right],$$

where $t > 0$ and $z \in \mathbb{R}$. The second equality is obtained by using $v_j = v_{j+q}$ for any integer j.

• (i)-2. The case $q = 4n + 1$ ($n \geqslant 1$). Since the rays of integrations in (3.10) with $m = 0, \ldots, n-1, 3n+1, \ldots, q-1$ (resp. $m = n, \ldots, 3n$) belong to R_+ (resp. R_-), we have

$$(*)_{(\mathrm{i})-2} \quad u_B^0(t,z) = \frac{1}{(qt)^{1/q}} \left[\int_0^{+\infty} \varphi(z+\zeta) \left\{ v_{n+1} + \cdots + v_{3n} \right\} (X) d\zeta \right.$$
$$\left. + \int_0^{-\infty} \varphi(z+\zeta) \left\{ v_1 + \cdots + v_n + v_{3n+1} + \cdots + v_q \right\} (X) d\zeta \right],$$

where $t > 0$ and $z \in \mathbb{R}$.

• (ii)-1. The case $q = 4n - 2$ ($n \geqslant 2$). Since the rays of integrations in (3.10) with $m = 0, \ldots, n-1, 3n-1, \ldots, q-1$ (resp. $m = n, \ldots, 3n-2$) belong to R_+ (resp. R_-), we have

$$(*)_{(\mathrm{ii})-1} \quad u_B^0(t,z) = \frac{1}{(qt)^{1/q}} \left[\int_0^{+\infty} \varphi(z+\zeta) \left\{ v_{n+1} + \cdots + v_{3n-1} \right\} (X) d\zeta \right.$$
$$\left. + \int_0^{-\infty} \varphi(z+\zeta) \left\{ v_1 + \cdots + v_n + v_{3n} + \cdots + v_q \right\} (X) d\zeta \right],$$

where $t > 0$ and $z \in \mathbb{R}$.

• (ii)-2. The case $q = 4n - 1$ ($n \geqslant 1$). Since the rays of integrations in (3.10) with $m = 0, \ldots, n-1, 3n, \ldots, q-1$ (resp. $m = n, \ldots, 3n-1$) belong to R_+ (resp. R_-), we have

$$(*)_{(\mathrm{ii})-2} \quad u_B^0(t,z) = \frac{1}{(qt)^{1/q}} \left[\int_0^{+\infty} \varphi(z+\zeta) \left\{ v_{n+1} + \cdots + v_{3n-1} \right\} (X) d\zeta \right.$$
$$\left. + \int_0^{-\infty} \varphi(z+\zeta) \left\{ v_1 + \cdots + v_n + v_{3n} + \cdots + v_q \right\} (X) d\zeta \right],$$

where $t > 0$ and $z \in \mathbb{R}$.

Therefore, by inserting the functional equality $(\mathbf{V2})_{(\mathrm{i})}$ or $(\mathbf{V2})_{(\mathrm{ii})}$ into the formulas $(*)_{(\mathrm{i})-1}$ and $(*)_{(\mathrm{i})-2}$ or the formulas $(*)_{(\mathrm{ii})-1}$ and $(*)_{(\mathrm{ii})-2}$, and by putting $z = x \in \mathbb{R}$, we obtain the desired result.

$$\begin{aligned} u_B^0(t,x) &= \frac{1}{(qt)^{1/q}} \int_{-\infty}^{+\infty} \varphi(x+\zeta) w\left(\frac{\zeta}{(qt)^{1/q}}\right) d\zeta \qquad (3.12) \\ &= u_c(t,x), \qquad t > 0,\ x \in \mathbb{R}. \end{aligned}$$

In order to complete the proof of Theorem 2.1, we shall prove the formula (3.11).

For that purpose, we use the asymptotic expansion of the G-function

$$\begin{aligned} & G_{0,q-1}^{q-1,0}\left(z \,\middle|\, \widehat{\boldsymbol{q}_q}/q\right) \qquad (3.13) \\ = \ & \frac{(2\pi)^{(q-2)/2}}{(q-1)^{1/2}} \exp\left\{-(q-1)z^{1/(q-1)}\right\} z^{1/2(q-1)} \left[1 + O\left(z^{-1/(q-1)}\right)\right], \\ & \text{as } |z| \longrightarrow \infty, \quad |\arg z| \leqslant q\pi - \sigma,\ \sigma > 0, \end{aligned}$$

with $q \geqslant 3$, which can be seen in [Luk, p. 179].

We recall the relationship between v_{2n} and the G-function

$$v_{2n}(X) = \frac{C_q}{X} G_{0,q-1}^{q-1,0}\left(\frac{X^q}{q^{q-1}\alpha} \,\middle|\, \widehat{\boldsymbol{q}_q}/q\right), \quad X = \frac{\zeta}{(qt)^{1/q}}, \qquad (3.14)$$

which follows from (1.14) and (3.6) which will be proved in the next section. Now, by the asymptotic expansion (3.13) and the functional equalities (**V1**) and (3.14), we have the following asymptotic expansions for $v_{2n-m}(X)$

$$\begin{aligned} & v_{2n-m}(X) \left(= v_{2n}(X\omega_q^{-m})\omega_q^{-m}\right) \qquad (3.15) \\ = \ & O\left(\exp\left\{-c\left(\frac{(X\omega_q^{-m})^q}{\alpha}\right)^{1/(q-1)}\right\} X^{q/2(q-1)-1}\right), \text{ as } |X| \to \infty, \end{aligned}$$

in the region $|\arg(X\omega_q^{-m}) - \arg\alpha/q| \leqslant \pi - \sigma'$ by some positive constants c and σ'.

Proof of (3.11). — Now, we give the proof of the formula (3.11) by dividing into four cases $q = 4n$, $4n-1$, $4n-2$ and $4n+1$.

• The case $q = 4n-2$. In this case, we note that $\alpha = 1$ and the rays of integrations with $m = 0, 1, \ldots, n-1, 3n-1, \ldots, q-1$ (resp. $m = n, \ldots, 3n-2$) belong to R_+ (resp. R_-) (see below Figure 4).

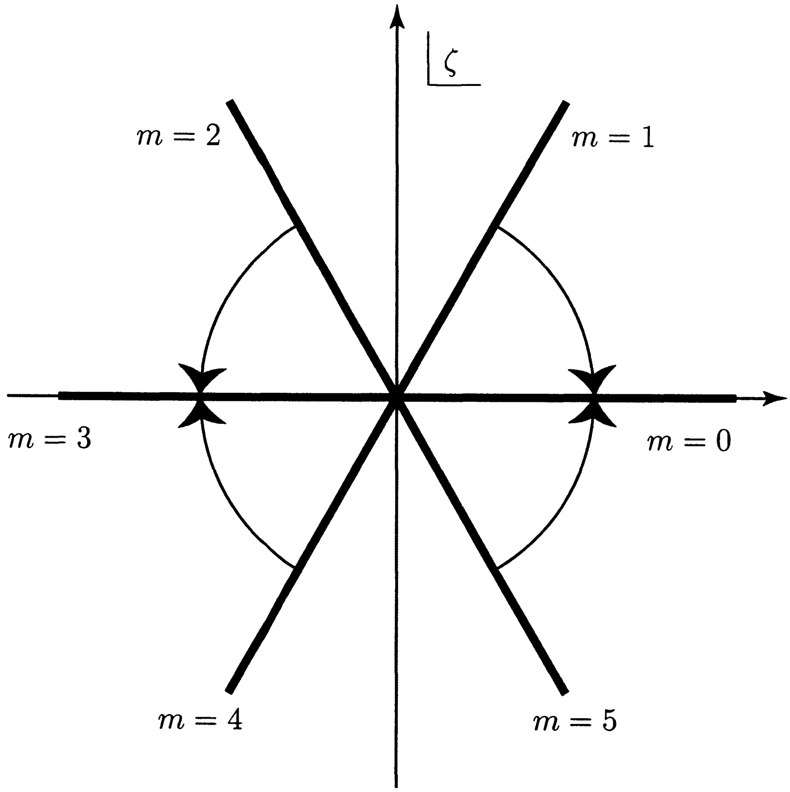

Figure 4. — Rays of integrations for the Borel sum in the case $q = 6$ $(n = 2)$

From the asymptotic estimate (3.15), each kernel function $v_{2n-m}(\zeta/(qt)^{1/q})$ in (3.10) for which the ray of integration belongs to R_+ (resp. R_-) has the exponential decreasing estimate of order $q/(q-1)$ as $|\zeta| \to \infty$ in the sector

$$\frac{-q+1+4m}{2q}\pi < \arg\zeta < \frac{q-1+4m}{2q}\pi,$$

which contains the positive (resp. negative) real axis. This enables us to change the ray of integration with the argument $2\pi m/q$ in R_+ (resp. R_-) into the positive (resp. negative) real axis under the conditions (I) for the Cauchy data $\varphi(z)$. The proof of the formula (3.11) in the case $q = 4n-2$ is complete.

• The case $q = 4n$. In this case, we note that $\alpha = -1$ and the rays of integrations with $m = 0, 1, \ldots, n-1, 3n, \ldots, q-1$ (resp. $m = n, \ldots, 3n-1$) belong to R_+ (resp. R_-).

By noticing these facts, the proof of the formula (3.11) in the case $q = 4n$ is done in the similar manner to the case $q = 4n - 2$.

• The case $q = 4n - 1$. In this case, we note that $\alpha = 1$ and the rays of integrations with $m = 0, 1, \ldots, n-1, 3n, \ldots, q-1$ (resp. $m = n, \ldots, 3n-1$) belong to R_+ (resp. R_-) (see below Figure 5).

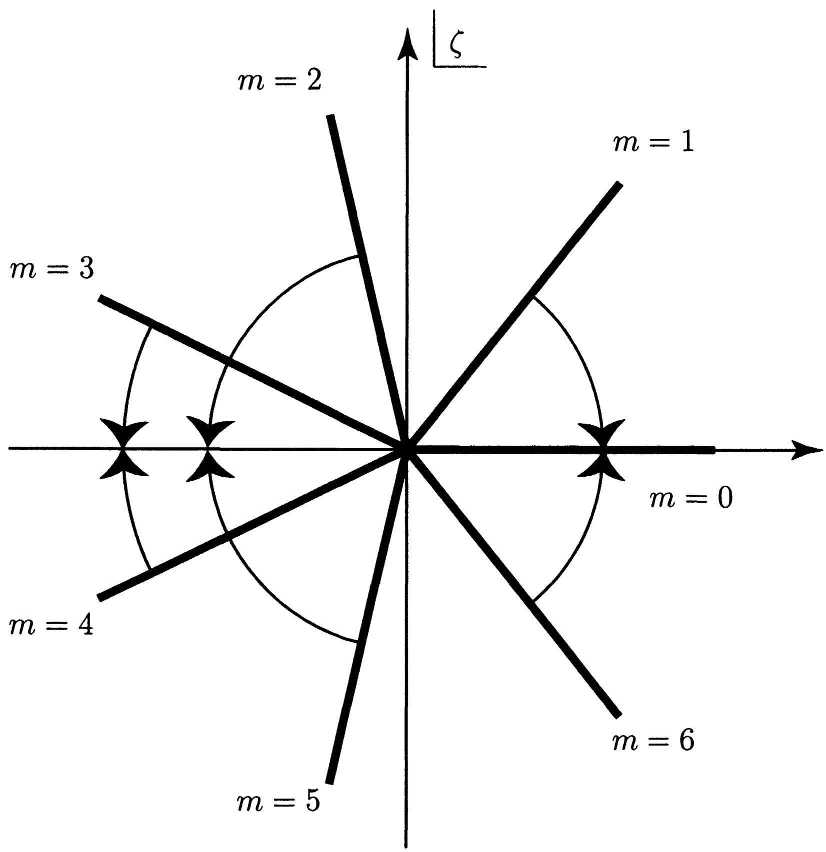

Figure 5. — Rays of integrations for the Borel sum in the case $q = 7$ $(n = 2)$

When $m = 0, \ldots, n-1, 3n, \ldots, q-1$ (resp. $m = n+1, \ldots, 3n-2$), each kernel function $v_{2n-m}(\zeta/(qt)^{1/q})$ in (3.10) has the exponential decreasing estimate of order $q/(q-1)$ as $|\zeta| \to \infty$ in the sector

$$\frac{-q+1+4m}{2q}\pi < \arg\zeta < \frac{q-1+4m}{2q}\pi,$$

which contains the positive (resp. negative) real axis. Therefore, for these m's the formula (3.11) follows by the same reasoning with the above.

We have to remark that the cases $m = n$ and $m = 3n-1$ are exceptional, because the functions $v_{2n-m}(\zeta/(qt)^{1/q})$ with $m = n$ and $m = 3n-1$ do not have the exponential decreasing estimate as $\zeta \to -\infty$ on the negative real axis. In deed, when $m = n$, the function $v_n(\zeta/(qt)^{1/q})$ has the following estimate, for small $\varepsilon > 0$

$$\left| v_n\left(\frac{\zeta}{(qt)^{1/q}}\right)\right| \tag{3.16}$$
$$\leqslant \begin{cases} C_\varepsilon \exp\left(-c_\varepsilon|\zeta|^{q/(q-1)}\right)|\zeta|^{q/2(q-1)-1}, & \pi/q < \arg\zeta \leqslant \pi-\varepsilon, \\ C_\varepsilon|\zeta|^{q/2(q-1)-1}, & \pi-\varepsilon \leqslant \arg\zeta \leqslant \pi. \end{cases}$$

When $m = 3n-1$, the function $v_{2n-(3n-1)}(\zeta/(qt)^{1/q}) = v_{3n}(\zeta/(qt)^{1/q})$ has the following estimate, for small $\varepsilon > 0$

$$\left| v_{3n}\left(\frac{\zeta}{(qt)^{1/q}}\right)\right| \tag{3.17}$$
$$\leqslant \begin{cases} C_\varepsilon \exp\left(-c_\varepsilon|\zeta|^{q/(q-1)}\right)|\zeta|^{q/2(q-1)-1}, & \pi+\varepsilon \leqslant \arg\zeta < \pi+\pi/q, \\ C_\varepsilon|\zeta|^{q/2(q-1)-1}, & \pi \leqslant \arg\zeta \leqslant \pi+\varepsilon. \end{cases}$$

Therefore if the Cauchy data $\varphi(z)$ is analytic in Δ_q and has the growth condition of exponential order at most $q/(q-1)$ there, then for a small fixed $\varepsilon > 0$, we have

$$\int_0^{\infty(2\pi n/q)} \varphi(z+\zeta)v_n(X)d\zeta = \int_0^{\infty(\pi-\varepsilon)} \varphi(z+\zeta)v_n(X)d\zeta, \tag{3.18}$$

$$\int_0^{\infty(2\pi(3n-1)/q)} \varphi(z+\zeta)v_{3n}(X)d\zeta = \int_0^{\infty(\pi+\varepsilon)} \varphi(z+\zeta)v_{3n}(X)d\zeta. \tag{3.19}$$

Further, if the Cauchy data $\varphi(z)$ has the polynomial decreasing condition (2.1) in the sector $S(\pi,\delta,\infty)$ with $\delta > 2\varepsilon$, then from the estimates (3.16) and (3.17), the absolute integrability on the negative real axis do hold for the integrals in the right hand side of (3.18) and (3.19), and we obtain the formula (3.11). This completes the proof of the formula (3.11) in the case $q=4n-1$.

• The case $q=4n+1$. In this case, we note that $\alpha=-1$ and the rays of integrations with $m=0,1,\ldots,n-1,3n+1,\ldots,q-1$ (resp. $m=n,\ldots,3n$) belong to R_+ (resp. R_-).

By noticing these facts, the proof of the formula (3.11) in the case $q=4n+1$ is done in the similar manner to the case $q=4n-1$.

From above observations, the proof of Theorem 2.1 is complete. □

Remark 3.3. — We remark the reason why the formula (1.1) works in a wider class than $\mathcal{S}$ for the Cauchy data $\varphi(x)$.

From the asymptotic estimates (3.15), (3.16) and (3.17), and the functional relations $(\mathbf{V2})_{(\mathrm{i})}$ and $(\mathbf{V2})_{(\mathrm{ii})}$, we have the following asymptotic estimates for $w(X)$.

When q is even,

$$\begin{aligned}
&|w(X)| \qquad\qquad (3.20)\\
=\ & |(v_{n+1}+\cdots+v_{3n-1})(X)| = |(v_1+\cdots+v_n+v_{3n}+\cdots+v_q)(X)|\\
\leqslant\ & C\exp\left\{-c|X|^{q/(q-1)}\right\}|X|^{q/2(q-1)-1} \quad \text{as } X\to\pm\infty,
\end{aligned}$$

by some positive constants C and c.

When q is odd,

$$\begin{aligned}
&|w(X)| \qquad\qquad (3.21)\\
=\ & |(v_{n+1}+\cdots+v_{3n-1})(X)| = |(v_1+\cdots+v_n+v_{3n}+\cdots+v_q)(X)|\\
\leqslant\ & \begin{cases} C\exp\left\{-c|X|^{q/(q-1)}\right\}|X|^{q/2(q-1)-1}, & \text{as } X\to+\infty,\\ C|X|^{q/2(q-1)-1}, & \text{as } X\to-\infty, \end{cases}
\end{aligned}$$

by some positive constants C and c.

Therefore, when q is even, the formula (1.1) works in a class such that the Cauchy data $\varphi(x)$ has the growth condition of exponential order at most $q/(q-1)$ as $|x|\to\infty$ on the real axis. When q is odd, the formula (1.1) works in a class such that the Cauchy data $\varphi(x)$ has the growth condition of exponential order at most $q/(q-1)$ as $x\to+\infty$ on the positive real axis and has the decreasing condition of polynomial order at most $q/2(q-1)+\lambda$ $(\lambda>0)$ as $x\to-\infty$ on the negative real axis.

4. Proof of the formula (3.6) in Proposition 3.2

In this section we shall prove the formula (3.6) in Proposition 3.2. For that purpose, it is enough to prove

$$\frac{C_q}{\zeta}\, G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \widehat{\boldsymbol{q}_q}/q\right) = \frac{1}{(q\tau)^{1/q}}\,\frac{1}{2\pi i}\int_{\gamma_{2n}} \exp\left[Xs + \alpha\frac{(-s)^q}{q}\right] ds, \quad (4.1)$$

where

$$Z_\alpha = \frac{1}{q^q\alpha}\frac{\zeta^q}{\tau}, \quad X = \frac{\zeta}{(q\tau)^{1/q}}, \quad (4.2)$$

$\alpha = 1$ when $q \equiv 2, 3 \pmod 4$, $\alpha = -1 = e^{\pi i}$ when $q \equiv 0, 1 \pmod 4$ and $C_q = 1/\prod_{j=1}^{q-1}\Gamma(j/q)$.

The following formula for the G-function can be seen in [Luk, p. 150].

$$z^\sigma G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\gamma}\end{matrix}\right) = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}\boldsymbol{\alpha}+\sigma\\ \boldsymbol{\gamma}+\sigma\end{matrix}\right), \quad (4.3)$$

where $\boldsymbol{\alpha} + \sigma = (\alpha_1 + \sigma, \alpha_2 + \sigma, \ldots, \alpha_p + \sigma)$. Then we have

$$\begin{aligned} &\frac{C_q}{\zeta}\, G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \widehat{\boldsymbol{q}_q}/q\right) \quad (4.4)\\ &= \frac{1}{(q\tau)^{1/q}}\,\frac{C_q}{q^{1-1/q}\alpha^{1/q}}\, Z_\alpha^{-1/q}\, G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \widehat{\boldsymbol{q}_q}/q\right)\\ &= \frac{1}{(q\tau)^{1/q}}\,\frac{C_q}{q^{1-1/q}\alpha^{1/q}}\, G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \widehat{\boldsymbol{q}_q}/q - 1/q\right), \quad Z_\alpha = \frac{1}{q^q\alpha}\frac{\zeta^q}{\tau}. \end{aligned}$$

Therefore it is enough to prove the following lemma.

LEMMA 4.1. —

$$\frac{C_q}{q^{1-1/q}\alpha^{1/q}}\, G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \widehat{\boldsymbol{q}_q}/q - 1/q\right) = \frac{1}{2\pi i}\int_{\gamma_{2n}} \exp\left[Xs + \alpha\frac{(-s)^q}{q}\right] ds, \quad (4.5)$$

where

$$Z_\alpha = \frac{1}{q^q\alpha}\frac{\zeta^q}{\tau}, \quad X = \frac{\zeta}{(q\tau)^{1/q}}.$$

We note the following relation between Z_α and X.

$$Z_\alpha = \frac{1}{q^{q-1}\alpha}X^q \quad \textit{or} \quad Z_\alpha^{1/q} = \frac{1}{q^{1-1/q}\alpha^{1/q}}X. \quad (4.6)$$

In order to prove the formula (4.5) in Lemma 4.1, we shall show that the power series expansions of both hand sides are the same ones. Precisely, we give the power series expansion at $Z_\alpha = 0$ of the left hand side and at $X = 0$ of the right hand side, respectively.

To do so, we give the definition of the generalized hypergeometric series.

The Generalized Hypergeometric Series. (cf. [Luk, p. 41]) For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q) \in \mathbb{C}^q$, we define

$$ {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\gamma}; z) = {}_pF_q\left(\begin{matrix} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{matrix} ; z \right) := \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n} \frac{z^n}{n!}, \tag{4.7} $$

where

$$ (\boldsymbol{\alpha})_n = \prod_{\ell=1}^{p} (\alpha_\ell)_n, \quad (\boldsymbol{\gamma})_n = \prod_{j=1}^{q} (\gamma_j)_n, \quad (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)} \quad (c \in \mathbb{C}). $$

By employing this terminology, we can obtain the following three formulas.

- When $q = 4n$ $(n \geqslant 1)$ or $q = 4n+1$ $(n \geqslant 1)$, we have

$$ \begin{aligned} &\frac{1}{2\pi i} \int_{\gamma_{2n}} \exp\left[Xs - \frac{(-s)^q}{q} \right] ds \\ &= \sum_{\ell=1}^{q-1} \frac{(-1)^{\ell-1} q^{\ell/q-1} e^{-\ell\pi i/q}}{(\ell-1)!\,\Gamma(1-\ell/q)} X^{\ell-1} {}_1F_{q-1}\left(\begin{matrix} 1 \\ (\widehat{\boldsymbol{q}}_q + \ell)/q \end{matrix} ; \frac{(-1)^q}{q^{q-1}} X^q \right). \end{aligned} \tag{4.8} $$

- When $q = 4n-2$ $(n \geqslant 2)$ or $q = 4n-1$ $(n \geqslant 1)$, we have

$$ \begin{aligned} &\frac{1}{2\pi i} \int_{\gamma_{2n}} \exp\left[Xs + \frac{(-s)^q}{q} \right] ds \\ &= \sum_{\ell=1}^{q-1} \frac{(-1)^{\ell-1} q^{\ell/q-1}}{(\ell-1)!\,\Gamma(1-\ell/q)} X^{\ell-1} {}_1F_{q-1}\left(\begin{matrix} 1 \\ (\widehat{\boldsymbol{q}}_q + \ell)/q \end{matrix} ; \frac{(-1)^{q-1}}{q^{q-1}} X^q \right). \end{aligned} \tag{4.9} $$

- We have

$$ \begin{aligned} &C_q G^{q-1,0}_{0,q-1}\left(Z_\alpha \,\middle|\, \begin{matrix} \\ \widehat{\boldsymbol{q}}_q/q - 1/q \end{matrix} \right) \\ &= \sum_{\ell=1}^{q-1} \frac{(-1)^{\ell-1} q^{\ell-1}}{(\ell-1)!\Gamma(1-\ell/q)} Z_\alpha^{(\ell-1)/q} {}_1F_{q-1}\left(\begin{matrix} 1 \\ (\widehat{\boldsymbol{q}}_q + \ell)/q \end{matrix} ; (-1)^{q-1} Z_\alpha \right). \end{aligned} \tag{4.10} $$

From these formulas, we can see that the power series expansions of both hand sides of the formula (4.5) in Lemma 4.1 are same ones by noticing the following relation

$$(-1)^{q-1}Z_\alpha = \frac{(-1)^{q-1}}{q^{q-1}\alpha}X^q, \quad {Z_\alpha}^{1/q} = \frac{1}{q^{1-1/q}\alpha^{1/q}}X.$$

In order to complete the proof of the formula (4.5) in Lemma 4.1, we prove the above three formulas (4.8), (4.9) and (4.10).

- We first give the proof of the formula (4.8).

In the case $q = 4n$ ($n \geqslant 1$) or $q = 4n+1$ ($n \geqslant 1$), we note $\alpha = -1$ by the assumption.

By expanding e^{Xs} in the integrand into its power series and by termwise integrating, we have

$$\frac{1}{2\pi i}\int_{\gamma_{2n}} \exp\left[Xs - \frac{(-s)^q}{q}\right] ds = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\int_{\gamma_{2n}} s^n \exp\left(-\frac{(-s)^q}{q}\right) ds.$$

We take the path γ_{2n} as a summation of two rays with the arguments $\pi - 2\pi/q$ and π. Then these integrals can be expressed in terms of the gamma integrals.

$$\begin{aligned}
&\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\int_{\gamma_{2n}} s^n \exp\left(-\frac{(-s)^q}{q}\right) ds \\
=\ &\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\left\{\int_0^{\infty(\pi-2\pi/q)} - \int_0^{\infty(\pi)}\right\} s^n \exp\left(-\frac{(-s)^q}{q}\right) ds \\
=\ &\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{X^n}{n!}\int_0^{\infty(0)} u^n \exp\left(-\frac{u^q}{q}\right) du \left\{e^{(n+1)(\pi-2\pi/q)i} - e^{(n+1)\pi i}\right\}.
\end{aligned}$$

Since

$$\int_0^{\infty(0)} u^n \exp\left(-\frac{u^q}{q}\right) du = q^{(n+1)/q-1}\Gamma\left(\frac{n+1}{q}\right),$$

$$\begin{aligned}
&e^{(n+1)(\pi-2\pi/q)i} - e^{(n+1)\pi i} \\
=\ &e^{(n+1)\pi i}\left\{e^{-2(n+1)\pi i/q} - 1\right\} \\
=\ &e^{n\pi i}e^{-(n+1)\pi i/q}\, 2i\sin\left(\frac{n+1}{q}\pi\right) \\
=\ &2i\,(-1)^n e^{-(n+1)\pi i/q}\frac{\pi}{\Gamma((n+1)/q)\Gamma(1-(n+1)/q)},
\end{aligned}$$

we have

$$\text{The left hand side of (4.8)} = \sum_{n\geqslant 0} \frac{q^{(n+1)/q-1}e^{-(n+1)\pi i/q}(-X)^n}{n!\Gamma(1-(n+1)/q)}.$$

Now, we divide the summation into q summations as follows.

$$\text{The left hand side of (4.8)} = \sum_{\ell=0}^{q-1}\left\{\sum_{k=0}^{\infty}\frac{q^{k-1+(\ell+1)/q}e^{-(k+(\ell+1)/q)\pi i}(-X)^{qk+\ell}}{(qk+\ell)!\Gamma(1-(\ell+1)/q-k)}\right\}$$

When $\ell = q-1$ in the above summation, we notice that $1/\Gamma(1-(\ell+1)/q-k) = 1/\Gamma(-k) = 0$. Then we have

$$\text{The left hand side of (4.8)} = \sum_{\ell=0}^{q-2}\left\{\sum_{k=0}^{\infty}\frac{q^{k-1+(\ell+1)/q}e^{-(k+(\ell+1)/q)\pi i}(-X)^{qk+\ell}}{(qk+\ell)!\Gamma(1-(\ell+1)/q-k)}\right\}$$

From the relations

$$(qk+\ell)! = q^{qk}\ell!\left(\frac{\ell+\boldsymbol{q}}{q}\right)_k,$$

$$\Gamma\left(1-\frac{\ell+1}{q}-k\right) = \frac{\Gamma(1-(\ell+1)/q)(-1)^k}{((\ell+1)/q)_k},$$

the left hand side of (4.8) is continued by

$$\begin{aligned}
&= \sum_{\ell=0}^{q-2}\frac{q^{(\ell+1)/q-1}e^{-(\ell+1)\pi i/q}}{\ell!\,\Gamma(1-(\ell+1)/q)}(-X)^{\ell}\sum_{k=0}^{\infty}\frac{(-X)^{qk}}{q^{(q-1)k}((\widehat{\boldsymbol{q}_1}+\ell)/q)_k}\\
&= \sum_{\ell=1}^{q-1}\frac{q^{\ell/q-1}e^{-\ell\pi i/q}}{(\ell-1)!\,\Gamma(1-\ell/q)}(-X)^{\ell-1}\sum_{k=0}^{\infty}\frac{(-X)^{qk}}{q^{(q-1)k}((\widehat{\boldsymbol{q}_q}+\ell)/q)_k}\\
&= \sum_{\ell=1}^{q-1}\frac{(-1)^{\ell-1}q^{\ell/q-1}e^{-\ell\pi i/q}}{(\ell-1)!\,\Gamma(1-\ell/q)}X^{\ell-1}{}_1F_{q-1}\left(\begin{matrix}1\\(\widehat{\boldsymbol{q}_q}+\ell)/q\end{matrix};\frac{(-1)^q}{q^{q-1}}X^q\right).
\end{aligned}$$

This completes the proof of the formula (4.8).

- We next give the proof of the formula (4.9). In the case $q = 4n-2$ $(n \geqslant 2)$ or $q = 4n-1$ $(n \geqslant 1)$, we note $\alpha = 1$ by the assumption.

By taking the path γ_{2n} as a summation of two rays with the arguments $\pi - \pi/q$ and $\pi + \pi/q$, we obtain the formula (4.9) in the similar way with the above procedure.

• At the end, we give the proof of the formula (4.10).

In order to do so, we employ the integral representation (1.12) of the G-function. The following expansion is obtained by calculating the residues of the left side of the path of integration $I = \{\mathrm{Re}\ s = \kappa; \kappa > 0\}$ in (1.12).

$$G_{0,q-1}^{q-1,0}\left(Z_\alpha \middle| \widehat{\boldsymbol{q}_q}/q - 1/q\right) = \frac{1}{2\pi i}\int_I \prod_{j=1}^{q-1} \Gamma\left(\frac{j-1}{q} + s\right) {Z_\alpha}^{-s} ds$$

$$= \sum_{\ell=1}^{q-1}\sum_{n\geqslant 0} \frac{(-1)^n}{n!} \prod_{j=1, j\neq \ell}^{q-1} \Gamma\left(\frac{j-\ell}{q} - n\right) {Z_\alpha}^{(\ell-1)/q+n}$$

$$= \sum_{\ell=1}^{q-1} \prod_{j=1, j\neq\ell}^{q-1} \Gamma\left(\frac{j-\ell}{q}\right) {Z_\alpha}^{(\ell-1)/q} {}_1F_{q-1}\left({1 \atop (\widehat{\boldsymbol{q}_q} + \ell)/q} ; (-1)^{q-1} Z_\alpha\right).$$

Therefore, the proof of the formula (4.10) is complete by noticing the following relation.

$$C_q \times \prod_{j=1, j\neq \ell}^{q-1} \Gamma\left(\frac{j-\ell}{q}\right) = \prod_{j=1, j\neq\ell}^{q-1} \Gamma\left(\frac{j-\ell}{q}\right) \Big/ \prod_{j=1}^{q-1} \Gamma\left(\frac{j}{q}\right)$$
$$= \frac{(-1)^{\ell-1} q^{\ell-1}}{(\ell-1)!} \frac{1}{\Gamma(1-\ell/q)},$$

which is obtained from the multiplication formula for the gamma function (cf. [Luk, p. 11])

$$\Gamma(mz) = (2\pi)^{-(m-1)/2} m^{mz-1/2} \prod_{j=0}^{m-1} \Gamma\left(z + \frac{j}{m}\right), \tag{4.11}$$

where $z + j/m \notin Z_{\leqslant 0} := \{0, -1, -2, \ldots\}$ $(j = 0, 1, \ldots, m-1)$.

From above observations, the proof of the formula (3.6) in Proposition 3.2 is complete. □

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