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Summability of power series in several variables, with applications to singular perturbation problems and partial differential equations[(*)]

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ABSTRACT. — We introduce and study a summability method of power series in several variables, and investigate applications to formal solutions of singular perturbation problems and partial differential equations. Doing so, we extend results of Lutz, Miyake and Schäfke, resp. Balser, for the complex heat equation to more general cases.

RÉSUMÉ. — Nous introduisons et considérons une méthode de sommabilité de séries de puissances en plusieures variables et nous donnons des applications aux solutions formelles de problèmes de perturbations singulières et aux équations aux dérivées partielles. Nous étendons ainsi des résultats de Lutz, Miyake et Schaefke, resp. Balser, pour l'équation de la chaleur variables complexes pour des cas plus généraux.

0. Introduction

Recently, work has been done to generalize the results from the theory of summability to power series in several variables: In papers by *Lutz, Miyake and Schäfke* [8] and *W. Balser* [2], the unique formal solution of a Cauchy problem for the complex heat equation has been shown to be k-summable if, and only if, the initial data has corresponding properties. For analogous results for a class of equations of parabolic type, see a manuscript of *Ichinobe and Miyake* [10]. In another article by *Balser and Miyake* [4] it has been clarified that such summability results are not so much due to an underlying partial differential equation, but only depend upon the fact that

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the coefficients in the formal solution satisfy a differential recursion relation of a certain (simple) form.

The results mentioned above, as well as corresponding ones in the theory of singular perturbations, can be viewed in the setting of power series in one variable with coefficients depending upon the remaining ones. As has been shown in [3], the general theory of multisummability carries over to this situation without difficulties. However, the following simple example suggests that an even more general approach is appropriate: Consider a series of the form

$$\hat{f}(z_1, \dots, z_m) = \sum_{n_1, \dots, n_m = 0}^{\infty} \Gamma(1 + s_1\, n_1 \cdots s_m\, n_m)\; z_1^{n_1} \cdots z_m^{n_m} \tag{0.1}$$

with $s_j \geqslant 0$. This series does not converge for any values $z_j \neq 0$, $1 \leqslant j \leqslant m$, except when all s_j vanish. If at least one of the s_j is rational, one can show that $\hat{f}$ satisfies a partial differential equation which we omit here. In view of the integral representation of the Gamma function, it is quite natural to define the function f, given by

$$f(z_1, \dots, z_m) = \int_0^{\infty} \mathrm{e}^{-x}\; \frac{1}{(1 - x^{s_1}\, z_1) \cdots (1 - x^{s_m}\, z_m)}\, dx,$$

as the sum of $\hat{f}$: Expanding the integrand into a power series in $z_1, \dots, z_m$ and integrating termwise, we obtain the series $\hat{f}$. In addition, f satisfies the same partial differential equation as $\hat{f}$ and is asymptotically equal to $\hat{f}$ in some polysector. In fact, we shall show later that this asymptotic is of a certain Gevrey type and determines f uniquely, due to the size of the polysector. More general series can be summed in an analogous way, and this is exactly what we shall investigate in this article: Consider any formal power series $\hat{f}$ in finitely many variables $z_1, \dots, z_m$. We then try to find a sum $f(z_1, \dots, z_m) = \int_0^\infty \mathrm{e}^{-x}\, g(x^{s_1}\, z_1, \dots, x^{s_m}\, z_m)\, dx$ with g holomorphic near the origin, chosen so that termwise integration of its power series gives $\hat{f}$. At first glance, this is a transformation affecting all variables at a time. However, introducing suitable new variables depending upon $s_1, \dots, s_m$, we shall see that this corresponds to an application of Borel and Laplace transform applied to a single variable, but *this single variable will not be one of the original variables* z_k, except when all but one s_j vanish.

1. Some basic notation

Throughout, let m be an arbitrarily fixed natural number. We shall assume $m \geqslant 2$, although most of what follows is correct, but well known, for

$m = 1$ as well. Let $\mathbb{S}$ stand for the universal covering surface of $\mathbb{C} \setminus \{0\}$, or in other words, for the Riemann surface of the logarithm. By $z = (z_1, \ldots, z_m)$ we denote an arbitrary point of $\mathbb{S}^m$, and by $\|z\|$ we mean the Euklidean norm of its projection into $\mathbb{C}^m$. Let $\mathbb{B}$ denote the set of $z \in \mathbb{S}^m$ with $\|z\| = 1$. The set $\mathbb{B}$ can be regarded as a metric space, with the metric locally induced by the Euklidean norm. However, note that we consider $\mathbb{B}$ as a subspace of $\mathbb{S}^m$, so that it is *not bounded with respect to this metric.*

For $r = (r_1, \ldots, r_m)$, with r_j either positive real numbers or $+\infty$, we consider *the polydisc* $D(r) = D(r_1) \times \ldots \times D(r_m) = \{z : |z_j| < r_j,\ 1 \leqslant j \leqslant m\}$. A *polysector* $S = S(d, \alpha, r)$, with r as above, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j > 0$, and $d = (d_1, \ldots, d_m) \in \mathbb{R}^m$, will be the Cartesian product of sectors $S_j = S(d_j, \alpha_j, r_j) = \{z_j : 0 < |z_j| < r_j, |d_j - \arg z_j| < \alpha_j/2\}$. If all $r_j = +\infty$, we shall also write $S(d, \alpha)$ for the corresponding polysector of infinite radius. Note that some or all of the numbers α_j may be larger than 2π; this is why we always consider polysectors in $\mathbb{S}^m$. However, we shall not always distinguish between a polydisc in $\mathbb{C}^m$ resp. $\mathbb{S}^m$.

By s we will always denote a vector $(s_1, \ldots, s_m)$ with $s_j \geqslant 0$, and we let $|s| = s_1 + \ldots + s_m$. In some cases, but not always, we shall restrict s further and require that $s_j > 0$ for every $j = 1, \ldots, m$, and we shall express this by writing $s > 0$, compared to $s \geqslant 0$ when some or even all s_j may vanish. Given $s > 0$, set $t = (z_1 \cdot \ldots \cdot z_m)^{1/|s|}$ and $u_j = t^{-s_j} z_j$, $1 \leqslant j \leqslant m$. Observe that in $\mathbb{S}^m$ the mapping $z \mapsto t$ is single-valued and holomorphic, and so are the mappings $z \mapsto u_j$. Moreover, note that $u_1 \cdot \ldots \cdot u_m = 1$, so that the $(n+1)$-dimensional vector $(t, u) = (t, u_1, \ldots, u_m)$ contains m *independent entries.* The mapping

$$z \longmapsto (t, u_1, \ldots, u_m) = (t, u)$$

is holomorphic and, due to the one relation eliminating one variable u_j, say: the last one, may be regarded as mapping $\mathbb{S}^m$ bijectively onto itself. So in what follows we shall freely go back and forth between the variables $z_1, \ldots, z_m$ and (t, u) to describe subsets of $\mathbb{S}^m$. In particular, we shall write $f(z) = f(z_1, \ldots, z_m)$ resp. $f(t, u) = f(t, u_1, \ldots, u_m)$ to denote one and the same function f, defined on some subset $G \subset \mathbb{S}^m$, either as depending on the variables $z_1, \ldots, z_m$ or on $t, u_1, \ldots, u_m$, always keeping in mind that the product of the u_j equals 1.

Given $s > 0$, a region G shall be called *an s-region, provided that it is an open and simply connected subset of a polysector in $\mathbb{S}^m$ satisfying the following condition:*

- For every $z = (z_1, \ldots, z_m) \in G$ and every real x with $0 < x \leqslant 1$, all points of the form $\zeta(x, z) = (x^{s_1} z_1, \ldots, x^{s_m} z_m)$ belong to G.

Note that every polysector is an s-region, for whatever vector s, but the converse does not hold. Suppose that, instead of the above condition, we have the following more restrictive one:

- For every $z = (z_1, \dots, z_m) \in G$ and every real x with $0 < x < +\infty$, all points of the form $\zeta(x, z) = (x^{s_1} z_1, \dots, x^{s_m} z_m)$ belong to G.

In this case, we shall call G *an s-region of infinite radius.* The above regions are easier to visualize using the variables (t, u): Replacing z by $\zeta(x, z)$ replaces t by xt, while the u_k stay fixed. Therefore, *an s-region G is characterized by the property that for fixed $u_1, \dots, u_m$ the projection of G onto the variable t is a sectorial region* in the sense of [1, 3]. *In case of G having infinite radius, this projection is a sector of infinite radius.* The relevance of these regions will become clear in the following section.

For any point z in an s-region G, there is a unique $x > 0$ so that the point $\zeta(x, z)$ has Euklidean norm 1. The set of all such points is a bounded, open[1] and simply connected subset of $\mathbb{B}$, denoted by $\mathcal{O}(G)$ and named *the opening* of the s-region G. Given any bounded, open and simply connected $\mathcal{O} \subset \mathbb{B}$, *there is a unique s-region G of infinite radius with $\mathcal{O}(G) = \mathcal{O}$.* Thus every such $\mathcal{O}$ will be called *an opening.*

Given an s-region G, we write $\overline{G}$ for its set-theoretic closure in $\mathbb{S}^m$. Note that in $\mathbb{S}^m$ this closure *does not contain points having one or several vanishing coordinates.* For a polysector S, observe that $\overline{S}$ may be unbounded, hence is *not* the analogue of a closed sector in the sense of [1, 3]. An s-region G_1 with $\overline{G_1} \subset G$ will be called *a proper subregion of* G, and we then write $G_1 \Subset G$.Observe that then $\mathcal{O}(G_1) \Subset \mathcal{O}(G)$, and conversely for any opening $\mathcal{O} \Subset \mathcal{O}(G)$ there exist $G_1 \Subset G$ with $\mathcal{O}(G_1) = \mathcal{O}$. If G is of infinite radius, we define $\mathcal{H}^{(s)}(G)$ to be the set of all f which are holomorphic in G and have the following property: *For every opening $\mathcal{O} \Subset \mathcal{O}(G)$ there exist constants $c, K > 0$ such that*

$$|f(\zeta(x,z))| \leqslant c\, \mathrm{e}^{K\,x} \qquad \forall\, z \in \mathcal{O},\ x > 0. \tag{1.1}$$

By $\mathbb{C}\,[[z]]$ we shall mean the differential algebra of formal power series in m variables $\hat{f}(z) = \sum_n f_n\, z^n$, with arbitrarily given complex coefficients $f_n = f_{n_1,\dots,n_m}$. We say that such a formal series *converges*, if we can find some radius r as above such that we have convergence in the polydisc $D(r)$. Note that then convergence is absolute, and uniform on every polydisc of

(1) Open in the subspace topology on $\mathbb{B}$. Also note that $\mathbb{B}$ is a subset of $\mathbb{S}^m$, hence boundedness is a non-trivial condition.

smaller radius, and the *sum* of $\hat{f}(z)$ is a holomorphic function in m variables, which we shall always denote by $f(z)$. It is easy to see that convergence of $\hat{f}(z)$ is equivalent to an estimate for the coefficients of the form $|f_n| \leqslant C\,K^{|n|}$, with suitably large constants C, K, independent of n, and $|n| = n_1 + \ldots + n_m$. The set of all convergent power series shall be denoted by $\mathbb{C}\{z\}$. If the coefficients f_n satisfy

$$|f_n| \leqslant C\,K^{|n|}\,\Gamma(1+s\,n)\,, \quad s\,n \;=\; \sum_{j=1}^{m} s_j\,n_j\,, \qquad \forall n \in \mathbb{N}_0^m,$$

with C, K as above and $s = (s_1, \ldots, s_n) \geqslant 0$, then we say that $\hat{f}(z)$ is *of Gevrey order* s, and write $\hat{f} \in \mathbb{C}\,[[z]]_s$. Every series $\hat{f} \in \mathbb{C}\,[[z]]_s$ *converges with respect to those* z_j *with* $s_j = 0$, in case there are any, in the following sense: Suppose for simplicity that $s_j = 0$ for $\mu + 1 \leqslant j \leqslant m$, and let $v = (z_1, \ldots, z_\mu)$, $w = (z_{\mu+1}, \ldots, z_m)$, $p = (n_1, \ldots, n_\mu)$, $q = (n_{\mu+1}, \ldots, n_m)$, hence $n = (p, q)$. Then $\hat{f}(z) = \sum_p f_p(w)\,v^p$, with $f_p(w) = \sum_q f_{(p,q)}\,w^q$, and all these series converge for w in a polydisc which is *independent of* p. *For this reason, it is without loss of generality when we shall study summability of power series* $\hat{f} \in \mathbb{C}\,[[z]]_s$ *only for cases where all* s_j *are positive.*

2. Integral operators of Laplace type

Let $s > 0$ be given, and let G be an s-region of infinite radius. For $f(z) \in \mathcal{H}^{(s)}(G)$ we define a function $g = \mathcal{L}^s f$ by the integral

$$g(z) = \int_0^\infty \mathrm{e}^{-x}\; f(x^{s_1} z_1, \ldots, x^{s_m} z_m)\; dx = \int_0^\infty \mathrm{e}^{-x}\; f(\zeta(x, z))\; dx. \qquad (2.1)$$

Consider an opening $\mathcal{O} \Subset \mathcal{O}(G)$. With K as in (1.1) and $w \in \mathcal{O}$, let $z = \zeta(y, w)$ with $0 < y < y_0 = K^{-1}$. These z obviously are an s-region which we denote by $G(\mathcal{O}, K)$. Since $\zeta(x, z) = \zeta(x\,y, w)$, we conclude from (1.1) that the integral (2.1) converges *absolutely and locally uniformly* in $G(\mathcal{O}, K)$. Making y_0 even smaller, we may turn the path of integration in (2.1) to become a ray $\arg x = \tau$ with $|\tau| < \pi/2$, and at the same time make the variables z_j move in the opposite direction. The integral (2.1) then converges in a region $G_\tau(y_0)$, whose opening $\mathcal{O}_\tau$ is the preimage of $\mathcal{O}$ under the mapping $z \mapsto \zeta(\mathrm{e}^{i\tau}, z)$. In this fashion we can see that $\mathcal{L}^s f$ becomes a holomorphic function in an s-region $\tilde{G}$, linked to G by the following condition:

- The opening $\mathcal{O}(\tilde{G})$ is the union of all $\mathcal{O}_\tau$ defined above, with arbitrary $\mathcal{O} \Subset \mathcal{O}(G)$ and $|\tau| < \pi/2$.

In the variables (t, u) the above description of $\tilde{G}$ says that for every fixed u the projection of $\tilde{G}$ onto the variable t is a sectorial region of opening more than π. If $m = 1$, a change of variable in (2.1) shows that $\mathcal{L}^s f$ coincides with Laplace transform of order $k = 1/s_j$ in the sense of [1, 3]. In general, we can rewrite (2.1) as saying that $g(t,u) = \int_0^\infty \mathrm{e}^{-x} f(xt,u)\,dx$. This shows that g equals the Laplace transform of order 1 of f with respect to the variable t. Using the inversion formula and substituting accordingly to return to the original variables, we find

$$f(z) = \frac{1}{2\pi i}\int_\gamma \mathrm{e}^x\, g(x^{-s_1} z_1, \ldots, x^{-s_m} z_m)\,\frac{dx}{x} \tag{2.2}$$

with γ as follows: From infinity along a ray $\arg x = -(\pi - \varepsilon)/2$ to a circle of radius $r > 0$ about the origin, along the circle to the ray $\arg x = (\pi + \varepsilon)/2$, and along that ray back to infinity. Observe that for $z \in G$ one can find a small $\varepsilon > 0$ and large r, so that the values $\zeta(x^{-1}, z)$ are in $\tilde{G}$ for every x on this path. Thus we may think of (2.1) to be a *generalization of Laplace transform to functions of several variables*, and (2.2) gives the inverse operator $\mathcal{B}^s$.

If we apply the operator $\mathcal{L}^s$ *formally*, i.e., termwise, to some formal power series $\hat{f}(z) = \sum_n f_n\, z^n$, we obtain the formal series $\hat{g}(z) = \sum_n f_n\, \Gamma(1 + s\,n)\, z^n$ which we regard as formal Laplace transform of $\hat{f}(z)$, writing $\hat{g} = \hat{\mathcal{L}}^s \hat{f}$. Analogously, the formal operator $\hat{\mathcal{B}}^s$ can be defined, being inverse to $\hat{\mathcal{L}}^s$. All the operators introduced give good sense even for $s = 0$, coinciding with identity operators.

3. Asymptotic expansions in several variables

In what follows, let $s > 0$ be fixed, and let G be an s-region. Given a function f, holomorphic in G, and a complex number $f(0)$, we write $f(z) \to f(0)$ as $z \to 0$ in G provided that for every proper subregion $G_1 \Subset G$ and every $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(0)| < \varepsilon$ whenever $z \in G_1$ with $\|z\| < \delta$. We say that $f(z)$ *is asymptotic to a formal power series* $\hat{f}(z) = \sum_n f_n\, z^n$, if for every multi-index n we have

$$f^{(n)}(z) = D^n f(z) \;\longrightarrow\; f_n\, n! \qquad \text{as } z \to 0 \text{ in } G, \tag{3.1}$$

with $n! = n_1! \cdots n_m!$ and $D^n = \partial_{z_1}^{n_1} \cdots \partial_{z_m}^{n_m}$. Given $\tilde{s} \geqslant 0$, we say that this asymptotic is of *Gevrey order* $\tilde{s}$, if for every *bounded* proper subregion $G_1 \Subset G$ there exist positive constants C, K so that

$$|D^n f(z)/n!\,| \leqslant C\, K^{|n|}\, \Gamma(1 + \tilde{s}\, n) \qquad \forall\, n \in \mathbb{N}_0^m,\; z \in G_1, \tag{3.2}$$

with $|n| = n_1 \cdots n_m$. This is a natural generalization of the corresponding notions for one variable to the several variable case and has been cosidered, e.g., by *J.-C. Tougeron* [13]. It is, however, different from the notion used by *H. Majima* in [9], resp. by *J. Mozo* in [12, 11]. Note that this estimate in particular implies $\hat{f} \in \mathbb{C}[[z]]_{\tilde{s}}$. We shall write $f(z) \cong_{\tilde{s}} \hat{f}(z)$ in G to indicate that $f(z)$ has $\hat{f}(z)$ as its Gevrey asymptotic of order $\tilde{s}$ in G. By $\mathcal{A}_{\tilde{s}}(G)$ we denote the set of all holomorphic functions f which have some Gevrey asymptotic of order $\tilde{s}$ in G, and $J\,f$ shall stand for the unique formal power series $\hat{f}$ which is the asymptotic expansion of f. Observe that the above statements all make good sense even for $\tilde{s} = 0$, in which case $f(z) \cong_0 \hat{f}(z)$ in G implies holomorphy of f at the origin, and $\hat{f}$ then converges to f for sufficiently small values of $z \in G$ – to see this, let z_0 vary within a proper subregion $G_1 \Subset G$ and observe that the m-dimensional Taylor series of $f(z)$ about z_0 then converges in a polydisc which *is independent of* z_0, so that we may let z_0 tend to the origin.

While in (3.2) we required an estimate on bounded proper subregions of G, it is necessary for investigating the asymptotic behavior of $\mathcal{L}^s f$ to assume an analogous estimate on proper subregions of infinite radius: Let $\mathcal{A}^s_{\tilde{s}}(G)$ stand for the set of functions which are holomorphic in G having the following property: *For every opening $\mathcal{O} \Subset \mathcal{O}(G)$ there exist constants $c, K > 0$ such that*

$$|f^{(n)}(\zeta(x,z))/n!\,| \leqslant c\,K^{|n|}\,\Gamma(1+\tilde{s}\,n)\;\mathrm{e}^{K\,x} \qquad \forall\; z \in \mathcal{O},\; n \in \mathbb{N}_0^m,\; x > 0. \tag{3.3}$$

In the one-variable situation, Cauchy's integral formula can be used to show that estimates (1.1) and (3.2) together imply (3.3); for $m \geqslant 2$, however, this is not necessarily so, as we learn from the following simple example: Let $f(z_1, z_2) = (1 - z_1 z_2)^{-1}$, which is holomorphic in $G = S \times S$, with $S = \{z : \; 0 < \arg z < \pi\}$. This G is polysector, hence an s-region for whatever $s > 0$. Clearly, f is holomorphic at the origin, hence satisfies (3.2) for $\tilde{s} = 0$. For $n = (0, \nu)$ we have $f^{(n)}(z_1, z_2) = \nu!\; z_1^{\nu}\,(1 - z_1 z_2)^{-\nu-1}$. For $\varepsilon > 0$ (small), the region $G_1 = S_1 \times S_1$, with $S_1 = \{z : \; \varepsilon < \arg z < \pi - \varepsilon\}$, is a proper subregion of G. Its opening contains vectors with abitrary small second component, so that $|f^{(n)}(\zeta(x,z))/n!\,|$ can be arbitrarily close to $x^{s_1\nu} z_1^{\nu}$. This term, however, cannot be estimated as in (3.3) with $\tilde{s} = 0$. On the other hand, if G is an s-region with an opening $\mathcal{O}(G)$ for which $z = (z_1, z_2) \in \mathcal{O}(G)$ implies $|z_j| > \varepsilon$, with some fixed $\varepsilon > 0$, then one can see that (3.3) indeed holds.

Assuming (3.3), we can now show that Laplace transform preserves Gevrey asymptotics but alters orders in a natural way:

THEOREM 3.1. — *Given two vectors $s > 0$and $\tilde{s} \geqslant 0$, define $\hat{s} = s + \tilde{s}$. Then for any s-region G of infinite radius and $f \in \mathcal{A}^s_{\tilde{s}}$, we have $\mathcal{L}^s f \in \mathcal{A}_{\hat{s}}(\tilde{G})$, for an s-region $\tilde{G}$ as described above. Moreover, $J \circ \mathcal{L}^s\, f = \hat{\mathcal{L}}^s \circ J\, f$ for every such f.*

Proof. — Consider any proper subregion $\tilde{G}_1 \Subset \tilde{G}$. Applying a compactness argument to the closure of its opening $\mathcal{O}(\tilde{G}_1)$, one can show that $\tilde{G}_1$ may be covered by finitely many s-regions G_{τ_k} on which we may represent $g = \mathcal{L}^s f$ by (2.1), integrating along some ray $\arg x = \tau_k$. For any multi-index n we then may calculate $D^n g$ by differentiation below the integral sign. Estimating the resulting integral on proper subregions $G_k \Subset G_{\tau_k}$, we obtain existence of C, K with

$$|D^n g(z)/n!| \leqslant C\, K^{|n|} \Gamma(1 + sn)\Gamma(1 + \tilde{s}n), \quad z \in G_k.$$

Using the Beta integral, one sees $\Gamma(1 + sn)\Gamma(1 + \tilde{s}n) \leqslant \Gamma(1 + \hat{s}n)$, and this can be used to complete the proof. □

An analogous result holds for the Borel operator $\mathcal{B}^s$: Let us say that an s-region G is *large* whenever some other s-region $\tilde{G}$ of infinite radius exists, so that for $z \in \tilde{G}$ we can find ε and r for which the path of integration γ used in (2.2) fits into G. Moreover, for $\tilde{s} \geqslant 0$ define $(\tilde{s} - s)_+$ to be the vector whose coordinates are the maximum of $\tilde{s}_j - s_j$ and 0.

THEOREM 3.2. — *Given two vectors $s > 0$ and $\tilde{s} \geqslant 0$, define $\hat{s} = (\tilde{s} - s)_+$. Then for any large s-region G and $f \in \mathcal{A}_{\tilde{s}}(G)$, we have $\mathcal{B}^s f \in \mathcal{A}_{\hat{s}}(\tilde{G})$, for an s-region $\tilde{G}$ of infinite radius. Moreover, $J \circ \mathcal{B}^s f = \hat{\mathcal{B}}^s \circ J f$ for every such f.*

The proof of this theorem is very much analogous to that of the previous one, resp. the corresponding result for $m = 1$, and is omitted here. As a consequence we mention that for $\tilde{s} = s$ we have $\hat{\mathcal{B}}^s \circ J f \in \mathbb{C}\{z\}$. This, together with the injectivity of $\mathcal{L}^s$ and $\mathcal{B}^s$, implies that for large s-regions G the mapping $J : \mathcal{A}_s(G) \longrightarrow \mathbb{C}[[z]]$ is injective. In other words:

- A function $f \in \mathcal{A}_s(G)$ is uniquely determined by its asymptotic whenever G is large in the above sense.

4. Partial asymptotics

In the previous section we have introduced asymptotic expansions in which all variables z_j approached the origin at a time. Sometimes it may be more natural to have only *some* variables do so while the others are fixed. However, for general regions, even for s-regions G, it may not be possible to do this without leaving G. For example, let u_j be restricted by $|u_1 - 1| < 1/2$, $u_1 u_2 = 1$, and set $z_j = t\, u_j$, for t in some sector S. This describes an s-region for $s = (1, 1)$ in which we cannot send z_1 or z_2 to 0 for any fixed value of the other variable. On the other hand, if G is a polysector, we can always do this. Therefore, we introduce the following terminology:

Let ϖ be a non-trivial subset of $\{1,\ldots,n\}$, and let ς denote its complement. For any point $z=(z_1,\ldots,z_m)\in\mathbb{S}^m$, let v be the vector of coordinates z_j for $j\in\varpi$ (in their natural order), and let w be defined analogously, but with ς instead of ϖ. We then can identify z with the pair (v,w). Using this notation, a region G will be named ϖ-suitable if two non-empty regions G_ϖ and G_ς of vectors v resp. w exist for which $G_\varpi\times G_\varsigma\subset G$, and if in addition G_ϖ has 0 as a boundary point. Given a multi-index n, we also can form two multi-indices p and q associated with the sets of indices ϖ and ς and identify n with (p,q). Let $\hat f_\varpi(z)=\sum_p f_p(w)\,v^p$ be a formal power series in the variable v, with coefficients depending upon w. For a ϖ-suitable region G and a function f, holomorphic in G, we then say that *f is ϖ-asymptotic to $\hat f_\varpi$ as $v\to 0$ in G provided that for every p as above $D_v^p f(v,w)\longrightarrow f_p(w)\,p!$ as $v\to 0$ in G_ϖ, with convergence being locally uniform for w in G_ς.* We shall also refer to ϖ as *the type* of asymptotic. If f has such an asymptotic, then for every q we conclude $D_w^q D_v^p f(v,w)\longrightarrow D_w^q f_p(w)$, and $D_w^q D_v^p$ can be identified with D^n, for $n=(p,q)$. Thus it makes sense to say that such an asymptotic expansion is of Gevery order $s\geqslant 0$ if (3.2) holds. It is not difficult to see that the above two theorems also hold for this *partial asymptotic expansion* of type ϖ, and we omit the proof here. Also observe that it is natural to consider $\varpi=\{1,\ldots,n\}$ as the type of the *complete asymptotic expansions* studied in the previous section. As a trivial but none the less important example, we mention the following: *If G is a polydisc and f is holomorphic in G, then f admits partial asymptotic expansions of all possible types.*

5. Summability in several variables

Given $s>0$ and an opening $\mathcal{O}$, we set $k=(1/s_1,\ldots,1/s_m)$ and let G denote the unique s-region of infinite radius with opening $\mathcal{O}$. Then we say that $\hat f(z)=\sum f_n\,z^n$ is *k-summable in direction $\mathcal{O}$* if the following two conditions hold:

1. The series $\hat f(z)$ belongs to $\mathbb{C}[[z]]_s$, so that $g(z)=\sum f_n\,z^n/\Gamma(1+sn)$ converges (in some polydisc about the origin of $\mathbb{C}^2$).
2. The function $g(z)$ can be continued into the region G, and its continuation is in $\mathcal{H}^{(s)}(G)$, so that $\mathcal{L}^s$ can be applied to obtain a function $f(z)$, which will be regarded as *the k-sum of $\hat f(z)$ in direction $\mathcal{O}$*, denoted as $f=\mathcal{S}_{k,\mathcal{O}}\,\hat f$. The set of all formal power series which are thus summable is denoted by $\mathbb{C}\{z\}_{k,\mathcal{O}}$.

This definition of k-summability shares many of the properties of the single-variable case, as we shall show now:

THEOREM 5.1. — *For arbitrary s and $\mathcal{O}$ as above, the set $\mathbb{C}\{z\}_{k,\mathcal{O}}$ is an algebra over $\mathbb{C}$, and the operator $\mathcal{S}_{k,\mathcal{O}}$ is a homomorphism. Moreover, $\mathbb{C}\{z\} \subset \mathbb{C}\{z\}_{k,\mathcal{O}}$.*

Proof. — Vector space property of $\mathbb{C}\{z\}_{k,\mathcal{O}}$ and linearity of $\mathcal{S}_{k,\mathcal{O}}$ follow directly from the definition. To deal with the product, let $\hat{f}_j \in \mathbb{C}\{z\}_{k,\mathcal{O}}$ be given, for $1 \leqslant j \leqslant 2$, and define $g_j(z)$ correspondingly. Setting

$$h(z) = \int_0^1 g_1(\zeta(x,z))\, g_2(\zeta(1-x,z))\, dx, \qquad z \in G,$$

we find that h is holomorphic in G and has a convergent power series representation at the origin which can be obtained by termwise integration of the power series of g_j. A straight-forward estimate, using $g_j \in \mathcal{H}^{(s)}(G)$, then implies $h \in \mathcal{H}^{(s)}(G)$. The function $g = \hat{\mathcal{B}}^s\,(\hat{f}_1 \hat{f}_2)$ is holomorphic near the origin and related to h by

$$g(z) = \frac{\partial}{\partial t}\, t\, h(z(t))_{|t=1} = \frac{1}{2\pi i} \oint \frac{t\, h(z(t))}{(t-1)^2}\, dt.$$

This shows that g is holomorphic in G, and estimating the integral we find $g \in \mathcal{H}^{(s)}(G)$.

For convergent $\hat{f}$, hence $|f_n| \leqslant C\, K^{|n|}$, the corresponding g is entire, and termwise integration of its power series expansion is justified, hence we have $\hat{f} \in \mathbb{C}\{z\}_{k,\mathcal{O}}$. $\square$

In what follows we consider the partial operators $\delta_j = z_j\,(d/dz_j)$ and show that k-summability and application of δ_j go together well:

THEOREM 5.2. — *For arbitrary s and $\mathcal{O}$ as above and every $j = 1, \dots, m$, the operators δ_j map $\mathbb{C}\{z\}_{k,\mathcal{O}}$ into itself, and*

$$(\delta_j \circ \mathcal{S}_{k,\mathcal{O}})\, \hat{f} = (\mathcal{S}_{k,\mathcal{O}} \circ \delta_j)\, \hat{f}, \qquad \forall \hat{f} \in \mathbb{C}\{z\}_{k,\mathcal{O}}.$$

Proof. — Given $\hat{f}$, let g be as in the definition of k-summability. Then the function $\delta_j\, g$ is the one corresponding to $\delta_j\, \hat{f}$, and using Cauchy's Formula one can show $\delta_j\, g \in \mathcal{H}^{(s)}(G)$. $\square$

For $s > 0$ and an opening $\mathcal{O}$, we shall say that a series $\hat{f}(z) \in \mathbb{C}\{z\}_{k,\mathcal{O}}$ is *strongly k-summable in direction* $\mathcal{O}$, provided that for every $\mathcal{O}_1 \Subset \mathcal{O}$ we can find constants c, K such that

$$|f^{(n)}(\zeta(x,z)| \leqslant c\,K^{|n|}\,\mathrm{e}^{K\,x} \qquad \forall n \in \mathbb{N}_0^m,\ x > 0,\ z \in \mathcal{O}_1. \tag{5.1}$$

As we have shown by an example in Section 3, this is a non-trivial condition. Using this terminology, the previous theorem has the following consequence:

THEOREM 5.3. — *In addition to the assumptions in the previous theorem, assume that $\hat{f}$ is strongly k-summable in direction $\mathcal{O}$. If the unique s-region G of opening $\mathcal{O}$ and infinite radius is of type ϖ, then $\mathcal{S}_{k,\mathcal{O}}\,\hat{f}$ has the partial asymptotic $\hat{f}_\varpi(z) = \sum_p f_p(w)\,v^p$, with $f_p = D_w^p\,\mathcal{S}_{k,\mathcal{O}}\,\hat{f}_\varpi$.*

Proof. — Observing that holomorphic functions have asymptotic expansions of arbitrary types, the proof follows from the analogue to Theorem 3.1 for asymptotics of type ϖ. □

As for the one-variable case, one can also define multisummability of power series in several variables, but we shall not consider this here.

6. Some applications

The following initial value problem for the complex heat equation has been investigated in [8, 2, 4]:

$$u_t = u_{zz}, \qquad u(0,z) = \hat{f}(z) = \sum_0^\infty f_n\,z^n,$$

with a power series $\hat{f}$ having positive radius of convergence. This problem is *formally well posed* in the sense that there is a unique formal power series in the two variable t and z, namely

$$\hat{u}(t,z) = \sum_{n=0}^\infty \frac{t^n}{n!}\,\hat{f}^{(2n)}(z) = \sum_{n,m=0}^\infty \frac{(m+2n)!}{n!\,m!}\,f_{m+2n}\,t^n\,z^m. \tag{6.1}$$

This power series solution also makes sense for divergent $\hat{f}$, hence it is natural to investigate its summability properties, assuming corresponding summability of $\hat{f}$. Some first, but not very satisfactory, result in this direction was obtained in [2, Section 2].

Before dealing with the above problem, we consider a related but slightly simpler situation arising for the following singular perturbation problem which was originally discussed by *J. Ecalle* [5, 6, 7]: The equation $\varepsilon\, x' = x - \hat f(z)$ has the formal solution

$$\hat x(z,\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m \hat f^{(m)}(z) = \sum_{n,m=0}^{\infty} \frac{(n+m)!}{n!} f_{n+m}\, z^n\, \varepsilon^m \tag{6.2}$$

for arbitrary $\hat f(z)$. The situation of convergent $\hat f$ leads to summability of $\hat x$, regarded as a power series in ε with holomorphic functions in z as coefficients; for a brief description of such results, see [3]. Here we study the following case:

- *For some* $0 < \kappa < 1$, *assume that* $\hat h(z) = (\hat{\mathcal{L}}^1 \hat f)(z) = \sum_0^\infty n!\, f_n\, z^n$ *is* κ*-summable in some direction* $d \in \mathbb{R}$.

Setting $\sigma = 1/\kappa - 1$, it follows from results in [3] that this assumption is equivalent to saying that

$$g(z) = \sum_0^\infty f_n \frac{z^n}{\Gamma(1+\sigma n)} \tag{6.3}$$

converges and the function so defined is holomorphic and of exponential growth at most κ in a sector $S = S(d,\alpha)$ of infinite radius, for some $\alpha > 0$, and the same then holds for the derivative g'. It will be convenient for estimates to follow to state the growth assumption as saying that for every $0 < \beta < \alpha$ there exist C, c so that for $z \in S(\beta, d)$

$$|g'(z)| \leqslant C \sum_1^\infty \frac{c^n\, |z|^{n-1}}{(n-1)!\,\Gamma(1+\sigma n)}\,, \quad |g(z)| \leqslant C \sum_0^\infty \frac{(c|z|)^n}{n!\,\Gamma(1+\sigma n)}\,; \tag{6.4}$$

to see that this indeed is equivalent to g and g' being of exponential growth at most κ, observe that the series

$$\sum_1^\infty \frac{c^n\, z^{n-1}}{(n-1)!\,\Gamma(1+\sigma n)} \qquad \text{and} \qquad \sum_0^\infty \frac{c^n\, z^n}{n!\,\Gamma(1+\sigma n)}$$

both define entire functions that are of exponential order κ and finite non-zero type, and hence can be estimated from above and below by $c \exp[K\,|z|^\kappa]$, for suitable constants c and K.

Under the above assumption, $\hat x$ is of Gevrey order $s = (\sigma, \sigma+1)$. Therefore, we introduce the function

$$y(z,\varepsilon) = (\hat{\mathcal{B}}^s \hat x)(z,\varepsilon) = \sum_{n,m=0}^{\infty} \frac{(n+m)!\, f_{n+m}}{n!\,\Gamma(1+\sigma(n+m)+m)}\, z^n\, \varepsilon^m .$$

This series converges, and the corresponding function has the following integral representation which may be found by termwise integration of the power series expansion of the integrand:

$$y(z,\varepsilon) = g(z) + \varepsilon \int_0^1 t^\sigma\, g'(t^\sigma[z+\varepsilon(1-t)])\, dt, \tag{6.5}$$

with $g(z)$ as in (6.3). For z and ε in S and $0 \leqslant t \leqslant 1$, we have $t^\sigma[z+\varepsilon(1-t)] \in S$, hence we conclude holomorphy of $y(z,\varepsilon)$ in the multisector $S \times S$. Using (6.4), one obtains for $z, \varepsilon \in S(d,\beta) \times S(d,\beta)$

$$|y(z,\varepsilon)| \leqslant C \sum_{n,m=0}^{\infty} \frac{c^{n+m}}{n!\,\Gamma(1+\sigma(n+m)+m)}\, |z|^n\, |\varepsilon|^m.$$

This implies the estimate we need to show *for $k = (\sigma^{-1}, (1+\sigma)^{-1})$ and $\mathcal{O}$ being the opening of the multisector $S \times S$ that $\hat{x}(z,\varepsilon)$ is k-summable in direction $\mathcal{O}$*. Since $\hat{h}(\varepsilon) = \hat{x}(0,\varepsilon)$, we also find that $\hat{h}(\varepsilon)$, formally regarded as a power series in two variables, is k-summable in direction $\mathcal{O}$. This, however, can be seen in this case to be equivalent to κ-summability in direction d (when regarding $\hat{h}$ as power series in one variable). *Thus we have that the above assumption is both necessary and sufficient for k-summability of $\hat{x}(z,\varepsilon)$.*

The treatment of the formal solution $\hat{u}(t,z)$ of the heat equation essentially follows the same lines, but is more involved. Here, we assume the following:

- *For some $0 < \kappa < 1/2$, let $\hat{h}(z) = \sum_0^\infty \Gamma(1+n/2)\, f_n\, z^n$ be κ-summable in directions d and $d+\pi$.*

With $\sigma = 1/\kappa - 1/2$, general results in [3] imply that this holds if, and only if, the series

$$g(z) = \sum_{n=0}^{\infty} f_n\, z^n / \Gamma(1/2 + \sigma\, n) \tag{6.6}$$

converges and the function so defined is holomorphic and of exponential growth at most κ in the sectors $S_+ = S(d,\alpha)$ and $S_- = S(d+\pi,\alpha)$, for some small positive α. In particular, this assumption implies that $\hat{f}$ is at most of Gevrey order $\sigma > 0$. Hence, the series $\hat{u}(t,z)$ is of Gevrey order $s = (2\sigma+1, \sigma)$, so we here consider the function

$$v(t,z) = \sum_{n,m=0}^{\infty} \frac{(2n+m)!\, f_{2n+m}}{n!\, m!\, \Gamma(1+\sigma(2n+m)+n)}\, t^n\, z^m.$$

To discuss holomorphic continuation of $v(t,z)$, we assume that $z \in S_+$ and $t \in S = S(2d, 2\alpha)$, interpreting $\sqrt{t}$ as being in S_+. With g as in (6.6), one checks by termwise integration (for sufficiently small values of $|z|, |t|$) that $2\sqrt{\pi}\, v(t,z) = v_+(t,z) + v_-(t,z)$, where

$$v_\pm(t,z) = \int_0^1 \frac{g\big(u^\sigma[z \pm 2\sqrt{(1-u)t}\,]\big)}{\sqrt{u(1-u)}}\, du. \tag{6.7}$$

The points $u^\sigma(z + 2\sqrt{(1-u)t})$ always are in S_+, so holomorphy of $v_+(t,z)$ in $S \times S_+$ follows from the integral representation. For $v_-(t,z)$ the same will, in general, be true only in a much smaller s-region $G_- \subset S \times S_+$ of infinite radius and an opening $\mathcal{O}_-$ which we now will construct: Recall that by assumption g is holomorphic in the union of the sectors $S_\pm$ and a disc about the origin of generally small radius; this union will be denoted by G from here on. Fixing a value $z \in S_+$, we shall show existence of $\delta > 0$ such that for all $t \in S$ with $|z|^2 + |t|^2 = 1$ and $|2 \arg z - \arg t| < \delta$, the function $v_-(x^{2\sigma+1}t, x^\sigma z)$ can be continued with respect to x all along the positive real axis. To do so, first observe that (6.7) implies for sufficiently small $x > 0$

$$v_-(x^{2\sigma+1}t, x^\sigma z) = \int_0^x \frac{g(h_x(u))}{\sqrt{u(x-u)}}\, du, \quad h_x(u) = u^\sigma(z - 2\sqrt{(x-u)t}).$$

We would like best to use the same integral representation for arbitrary x, but for large $x > 0$ the points $h_x(u) = u^\sigma z\,(1 - w\sqrt{x-u})$, with $w = 2\sqrt{t}/z$, in general are outside of the region G of holomorphy of g when integrating along the real axis . This problem does *not* arise for $\arg t = 2 \arg z$, because then $h_x(u)$ stays on the straight line through $\sqrt{t}$ and z. Since the region G is open and $h_x(u)$ depends continuously upon t, we see that for arbitrarily fixed $x_0 > 0$ there exists a $\delta_{x_0} > 0$ such that for $|\arg t - 2 \arg z| < \delta_{x_0}$ the curve parametrized by $h_x(u)$, $0 \leqslant u \leqslant x$, still is in G, meaning that we can use the integral representation to continue $v_-(x^{2\sigma+1}t, x^\sigma z)$ with respect to x to the interval $0 \leqslant x \leqslant x_0$. For $x_0 \to \infty$, however, the value of δ_{x_0} may tend to zero. To avoid this, we shall for $x \geqslant x_0 := 4|w|^{-2}$ integrate along a curve in the u-plane from 0 to x that consists of two straight line segments, parametrized as $u_1(\tau) = \tau\,(x - w^{-2})$, resp. $u_2(\tau) = x - (1-\tau)\,w^{-2}$, both for $0 \leqslant \tau \leqslant 1$. For the moment, let $\delta = \delta_{x_0}$, and let t be such that $|2 \arg z - \arg t| = |\arg w| < \delta$. Given $\varepsilon > 0$ (small), we see that we can make δ smaller so that $|\arg u_j(\tau)| \leqslant \varepsilon$ provided that $|\arg w| \leqslant \delta$, and $|\arg(w\sqrt{x - u_1(\tau)} - 1)| \leqslant \varepsilon$, while $1 - w\sqrt{x - u_2(\tau)}$ remains real, due to the choice of the path. For sufficiently small ε, this implies $h_x(u) \in G$ for every u on the path. Hence for the corresponding pair (t,z) we may represent $v_-(x^{2\sigma+1}t, x^\sigma z)$ by the above integral, for whatever value of $x > 0$.

Altogether, this shows existence of an opening $\mathcal{O}_-$ which contains arbitrary points $(t, z) \in \mathbb{B}$ with $|d - \arg z| < \alpha/2$ and $|\arg t - 2\arg z| < \delta$, with sufficiently small $\delta > 0$, depending upon z, so that $v_-(t, z)$ is holomorphic in the corresponding s-region G_- of infinite radius. In this region we can obviously let either one of the variables tend to 0 while keeping the other fixed, hence G_- is a region of all possible types ϖ.

To show k-summability of $\hat{u}(t, z)$ for $k = ((2\sigma + 1)^{-1}, \sigma^{-1})$, we have to estimate $v_\pm(t, z)$. To do so, let $G_{-,1} \Subset G_-$ be a proper subregion and use the integral representation (6.7) for $v_-(t, z)$, integrating along a path described above. Here, it is more natural to choose a parametrization $u(\tau)$ of the path with $0 \leqslant \tau \leqslant 1$, and doing so we observe that $|u(\tau)| \leqslant a\tau$, $|1 - u(\tau)| \leqslant a(1 - \tau)$, $|u'(\tau)| \leqslant a$ for some $a \geqslant 1$ and all τ where $u'(\tau)$ exists. For $v_+(t, z)$ we may then use the same notation by setting $u(\tau) = \tau$. We then observe that all points $u(\tau)^\sigma(z \pm 2\sqrt{(1 - u(\tau))t}$ remain within a proper subregion $G_1 \Subset G$, and there we have by assumption that $g(z)$ is of exponential growth not more than κ. This fact can be expressed as saying

$$|g(z)| \leqslant C \sum_{m=0}^{\infty} \frac{\Gamma(1 + m/2)}{m!\, \Gamma(1/2 + \sigma m)} (c|z|)^m,$$

with suitably large C, c. Estimating the integral accordingly, termwise integration leads to an estimate of the form

$$|v_\pm(t, z)| \leqslant C \sum_{n,m=0}^{\infty} (c_1|t|)^{n/2} (c_2|z|)^m \frac{\Gamma(1 + (m + n)/2)\, \Gamma((1 + n)/2)}{n!\, m!\, \Gamma(1 + \sigma(n + m) + n/2)},$$

which implies a similar estimate for $v(t, z)$. Estimating $|t|^{1/2}$ by 1 resp. $|t|$, for $|t| \leqslant 1$ resp. $|t| > 1$ and using Stirling's formula, one then can show

$$|v(t, z)| \leqslant C \sum_{n,m=0}^{\infty} \frac{(c_1|t|)^n (c_2|z|)^m}{\Gamma(1 + m/2)\, \Gamma(1 + \sigma(2n + m) + n)},$$

with different constants C, c_j. This estimate is sufficient to show summability of $\hat{u}(t, z)$ as stated above. As for the first example, one can again show that the above assumption on $\hat{f}$ is also necessary for k-summability of $\hat{u}(t, z)$. Also observe that the proof for $\sigma = 0$ may be simplified, and the result obtained coincides with the one proved in [8].

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